VOLUME 84 NO. ST8
DECEMBER 1958

JOURNAL of the

Structural Division

PROCEEDINGS OF THE

AMERICAN SOCIETY
OF CIVIL ENGINEERS

BASIC REQUIREMENTS FOR MANUSCRIPTS

This Journal represents an effort by the Society to deliver information to the reader with the greatest possible speed. To this end the material herein has none of the usual editing required in more formal publications.

Original papers and discussions of current papers should be submitted to the Manager of Technical Publications, ASCE. Authors should indicate the technical division to which the paper should be referred. The final date on which a discussion should reach the Society is given as a footnote with each paper. Those who are planning to submit material will expedite the review and publication procedures by complying with the following basic requirements:

1. Titles should have a length not exceeding 50 characters and spaces.

2. A 50-word summary should accompany the paper.

3. The manuscript (a ribbon copy and two copies) should be double-spaced on one side of 8½-in. by 11-in. paper. Papers that were originally prepared for oral presentation must be rewritten into the third person before being submitted.

4. The author's full name, Society membership grade, and footnote reference stating present employment should appear on the first page of the paper.

5. Mathematics are reproduced directly from the copy that is submitted. Because of this, it is necessary that capital letters be drawn, in black ink, ⅛-in. high (with all other symbols and characters in the proportions dictated by standard drafting practice) and that no line of mathematics be longer than 6½-in. Ribbon copies of typed equations may be used but they will be proportionately smaller on the printed version.

6. Tables should be typed (ribbon copies) on one side of 8½-in. by 11-in. paper with a 6½-in. by 10½-in. invisible frame. Small tables should be grouped within this frame. Specific reference and explanation should be made in the text for each table.

7. Illustrations should be drawn in black ink on one side of 8½-in. by 11-in. paper within an invisible frame that measures 6½-in. by 10½-in.; the caption should also be included within the frame. Because illustrations will be reduced to 69% of the original size, the capital letters should be ⅛-in. high. Photographs should be submitted as glossy prints in a size that is less than 6½-in. by 10½-in. Explanations and descriptions should be made within the text for each illustration.

8. Papers should average about 12,000 words in length and should be no longer than 18,000 words. As an approximation, each full page of typed text, table, or illustration is the equivalent of 300 words.

Further information concerning the preparation of technical papers is contained in the "Technical Publications Handbook" which can be obtained from the Society.

This Journal is published monthly by the American Society of Civil Engineers. Publication office is at 2500 South State Street, Ann Arbor, Michigan. Editorial and General Offices are at 33 West 39 Street, New York 18, New York. $4.00 of a member's dues are applied as a subscription to this Journal. Second-class postage paid at Ann Arbor, Michigan.

HY,ST.

Journal of the
STRUCTURAL DIVISION
Proceedings of the American Society of Civil Engineers

CONTENTS

December, 1958

Papers

Paper 1865 December, 1958 ST 8

Journal of the
STRUCTURAL DIVISION
Proceedings of the American Society of Civil Engineers

BIAXIALLY LOADED REINFORCED CONCRETE COLUMNS

Kuang-Han Chu,[1] A. M. ASCE and Algis Pabarcius,[2] J. M. ASCE
(Proc. Paper 1865)

SYNOPSIS

A numerical procedure is developed to determine the actual stress and strain distribution of a given or tentatively selected reinforced concrete section subjected to given compressive axial load and bending moments in both directions about principal axes. The investigation is based on the ultimate strength theories of Jensen and Hognestad.

INTRODUCTION

The theory of ultimate design has been adopted as an alternate design procedure by the 1956 ACI Building Code(1) as a result of the Joint Committee Report of ACI (American Concrete Institute) and ASCE(2) (American Society of Civil Engineers).

Design procedures for sections subjected to different loading conditions have been published in the report and in the paper by Whitney and Cohen.(3) However, no procedure was given in this paper for design and investigation of reinforced concrete sections subjected to compression and biaxial bending. A procedure was developed by Au(4) for designing rectangular sections. This procedure involves several simplifying assumptions, and has its limitations. (See later discussions).

The purpose of this paper is to develop such a method which would be applicable to sections of any shape (not necessarily rectangular). The main difficulties in developing the procedure are three-fold.

Note: Discussion open until May 1, 1959. To extend the closing date one month, a written request must be filed with the Executive Secretary, ASCE. Paper 1865 is part of the copyrighted Journal of the Structural Division, Proceedings of the American Society of Civil Engineers, Vol. 84, No. ST 8, December, 1958.

1. Assoc. Prof. Civ. Eng. Dept., Illinois Institute of Technology, Chicago, Ill.
2. Instructor, Civ. Eng. Dept., Univ. of Illinois, Chicago Division, Chicago, Ill.

(a) The simplified Whitney's theory using a rectangular stress block could not be applied directly to these sections without further verification or modification.
(b) The ultimate strain (not stress) of the concrete must be considered.
(c) Different stress-strain relationships in concrete and steel must be considered.

It is generally not difficult to find the ultimate capacity (due to axial load and moments) for a given reinforced concrete section, if the position of the neutral axis and the stress distribution are assumed or known for that section. However, in practice, the procedure must be reversed. Load factors are applied to the working loads and sections are designed to support that loading. A section is tentatively selected and it is required to find the actual stress and strain distribution. The ultimate capacity of the section is reached when the stress distribution corresponding to one of the modes of failure is obtained.

A procedure is developed in this paper to determine the actual stress and strain distribution more precisely and directly for reinforced concrete sections subjected to given direct compressive load and biaxial bending.

Notations

f	= stress
f_p	= plastic stress
f_e	= fictitious stress assuming elastic section
f_y	= yield point stress of steel
f'_c	= ultimate strength of concrete
f''_c	= $0.85f'_c$
ϵ_{max}	= maximum strain in concrete
ϵ_o	= elastic strain of concrete
ϵ_u	= ultimate strain of concrete
ϵ_{yp}	= yield point strain of steel
E_s	= steel modulus of elasticity
E_c	= concrete modulus of elasticity
n	= E_s/E_c
A	= area
A_c	= concrete area
A_s	= steel area
A_E	= area of elastic portion
A_{SE}, A_{CE}	= area of elastic portion of steel and concrete respectively
A_P	= area of plastic portion
A_{SP}, A_{CP}	= area of plastic portion of steel and concrete respectively

A_{CC} = area of cracked portion of concrete

P = axial force

P' = axial force taken by the plastic portion

P" = axial force taken by the elastic portion

M = bending moment

M' = moment taken by the plastic portion

M" = moment taken by the elastic portion

x, y = coordinates

M_x = moment about x axis

M_y = moment about y axis

I = moment of inertia

T = I about a centroidal axis

I_x = I about x axis

I_y = I about y axis

I_{xy} = product of inertia about x and y axes

M_o, I_o = moment and moment of inertia, respectively, referred to the centroid of the elastic portion

k' = ratio of d_1 to d_2 where

d_1 = distance from the extreme compressive corner to the inner edge of the stress block represented by a right prism whose centroid coincides with the centroid of the actual stress block

d_2 = distance from the extreme corner in compression to the neutral axis

d = distance from the compression fiber to the tension steel for rectangular sections in uniaxial bending

Modes of Failure and Stress Block

A reinforced concrete member subject to compression and flexure may have the following basic modes of failure.

1. Tension failure is caused by the yielding of the tensile reinforcing, thus causing the section to crack on the tension side. Further yielding of the steel will reduce the area of the concrete in compression to the extent that compressive failure will occur.
2. Compression failure can be described as failure of concrete in compression, before the full yield stress in the tension steel is developed. It will be caused by excessive compressive strain in the concrete.
3. A balanced condition will exist when the yielding of the steel and compression failure of the concrete will occur at the same instant.

The equations for any of the mentioned modes of failure can be derived after assuming a stress distribution in the concrete and considering the equations of equilibrium.

An accurate prediction of the ultimate strength of the section can be made if the assumed stress distribution for the concrete in the compression zone agrees with the actual inelastic stress-strain relationship for concrete.

It has been recognized that the inelastic stress distribution in the concrete compression zone of flexural members is of the shape shown in Fig. 1.

One important fact should be pointed out. It has been found that it is possible to develop an ultimate strain in flexure which is greater than the strain corresponding to the maximum stress developed in the compressive tests of plain concrete specimens. This compressive stress distribution is quite difficult to express in mathematical terms.

Various assumptions concerning this compressive stress distribution have been made in the earlier inelastic theories. Extensive illustrations of such theories have been compiled by Mr. Hognestad(5) and have since that time appeared in various other publications.(2,6)

The most extensive recent studies concerning this stress distribution have been made by Messrs. Hognestad,(5) Whitney(7) and Jensen.(8) The illustrations of their assumptions appear in Fig. 2.

As seen from these figures Hognestad and Jensen make an attempt to use a compressive stress distribution which would closely approximate the actual condition.

Whitney on the other hand uses a rectangular distribution as a mathematical device to approximate the effect of true distribution.

The Joint Committee Report(2) recognizes all three methods. It even suggests that any compressive stress distribution diagram may be used if the computed strengths agree with the results obtained from tests.

Of the above three methods, Whitney's method has the advantage of simplicity. However, to apply his method to nonrectangular sections or to sections subjected to biaxial bending, it is necessary to find a stress block represented by a right prism with a height of 0.85 f'_c which will give (a) the same compressive force and (b) the same point of application as the force given by the actual stress distribution. Or, in other words, it is necessary to

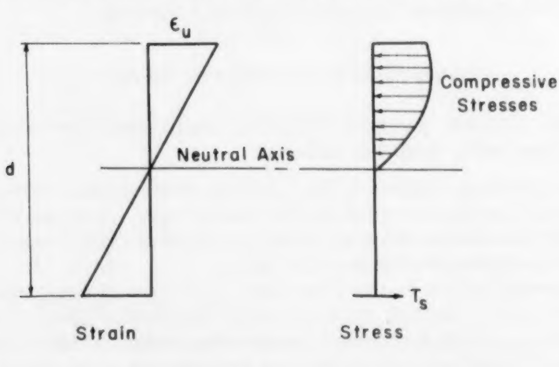


Fig. 1. Actual Stress Distribution

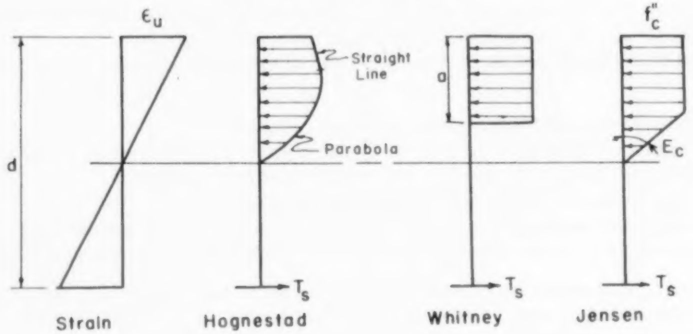


Fig. 2. Strain and Various Assumed Stress Distributions for Flexural Analysis

know (a) the ratio k' of the distance from the extreme corner in compression (or edge) to the inner edge of the base of this stress block to the distance from the same corner to the neutral axis and (b) the orientation of this edge with respect to the neutral axis.

Theoretically, this task is virtually impossible except for the very simple case of rectangular sections subjected to uniaxial bending. In this case, the inner edge of the stress block has to be parallel to the neutral axis and the distance ratio k' for $f'_c = 3{,}000$ psi is approximately equal to 0.85. For other cases, theoretically, if the centroid of the stress block of the right prism is made to coincide with the centroid of the volume of the actual stress distribution, the height of the stress prism will be different from $0.85f'_c$. On the other hand, if the height of the stress prism is assumed to be $0.85f'_c$, the centroid of its base will be different from that of the actual distribution. The above implies three equilibrium equations which will determine three unknowns, namely: the height of the prism and two dimensions of the base for the two coordinates of the centroid. Consequently, (a) the inner edge of the prism is not necessarily parallel to the neutral axis and (b) the ratio k' is not necessarily constant. Furthermore, even if approximate constants can be found for one type of section, new constants have to be found for other type of sections.

For the case of rectangular sections, fortunately, the above difficulties have been overcome. It was found that if the centroid of the stress prism is made to coincide with the centroid of the trapezoidal stress volume: (a) the height of the prism would be approximately equal to $0.85f'_c$; (b) the inner edge of the base of the prism would be approximately parallel to the neutral axis; and (c) the distance ratio k' would be approximately equal to 0.85. These points, however, were not realized until the numerical example in this paper had been worked out and a detail check was made on Au's method. Nevertheless, Au's method has the following limitations: (a) it could only determine the maximum strain indirectly, which involves the approximation of k' and of the direction of the neutral axis; (b) it could not be applied directly to sections other than rectangular without further verification or modification of the above approximations; (c) his assumption for locating the point of application of the resultant tension of the steel area is open to criticism; (d) his method is not

applicable in finding actual strain or stress distribution for sections carrying loads either less or more than its ultimate strength.

The method developed in this paper, however, has the following objectives in mind:

(a) It should be applicable to any kind of section.
(b) It would be adaptable to a more general type of stress-strain curve consisting of an elastic portion and a nonelastic portion.
(c) It would obtain the maximum strains directly.
(d) It would be applicable also for finding the strain or stress distribution for sections subjected to loads above or below their ultimate strength.

The numerical procedure is somewhat cumbersome. However, with the rapid development of electronic computer techniques it is hoped that the designer will be spared this drudgery. The trapezoidal distribution is assumed to simplify the numerical work.

Basic Assumptions

Summary of Assumptions

The development of the design procedure is based on the following assumptions.

1. Plane sections normal to the neutral axis remain plane after bending.
2. Tensile strength in concrete is neglected.
3. After the elastic stresses are exceeded, the stress distribution becomes trapezoidal in shape.
4. Maximum fiber stress in concrete f_c'' does not exceed $0.85f_c'$.
5. Maximum fiber stress in steel does not exceed f_y.

Stress Strain Relationship

The trapezoidal stress distribution was assumed in concrete and steel as shown in Fig. 3, to simplify the numerical work.

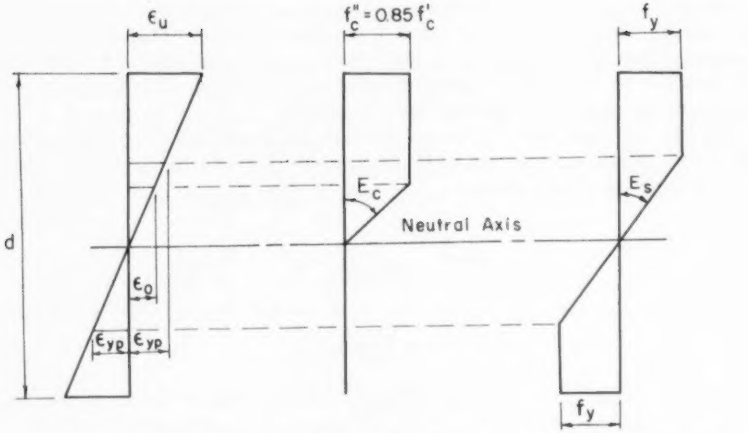
The method developed in this study can be used with any shape of stress-strain curve, if the curve has an elastic portion (see Fig. 3).

It has been found that the stress developed in flexure is only equivalent to 85 per cent of the cylinder strength. Thus, $f_c'' = 0.85f_c'$. Using the Joint Committee Report recommendations, the modulus of elasticity for concrete is taken as $E_c = 1{,}000f_c'$ and the ultimate strain $\epsilon_u = 0.003$. This means that the concrete will fail in compression when strained to that amount. From Fig. 3, it can be seen that the maximum strain at which the concrete remains elastic is $\epsilon_o = \dfrac{.85f_c}{E_c}$.

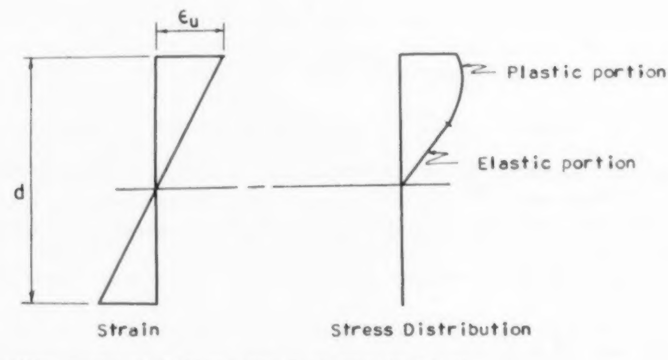
The stress distribution diagram for reinforcing steel is in general of trapezoidal shape for the elastoplastic case, with constant portion having the yield-point stress.

Further Discussion of Modes of Failure

The basic modes of failure for reinforced concrete in ultimate design are (a) tension failure, (b) compressive failure, and (c) a balanced condition



(a) Trapezoidal Stress-Strain Curves



(b) Other Types of Stress-Strain Curves to which the Method is Applicable

Fig. 3. Stress-Strain Relationships

between (a) and (b). Their meaning has been described earlier in the paper.

For members subjected to symmetrical bending, it was usually assumed that tension failure takes place as soon as the tension steel starts to yield. This is not necessarily true for unsymmetrical bending. Even after the reinforcing bar farthest from the neutral axis starts to yield, the stresses in the other tension bars still remain below the yield point stress. According to the definition of tension failure, this still does not constitute a failure. The member is capable of taking some additional loading. This condition will exist until the compressive strain in the concrete exceeds the ultimate value.

Of course, it may be stated that after some tension steel started to yield, the member had cracked and lost its functional value. Therefore, it would be important for the designer to decide what constitutes a failure: cracking of the member or inability to support additional loading. The method used in this paper can be used for checking both criteria.

Fig. 4a illustrates the stress block for modes of failure based on the yielding of the extreme steel bar in tension, while Fig. 4b shows the stress block for the mode of final collapse.

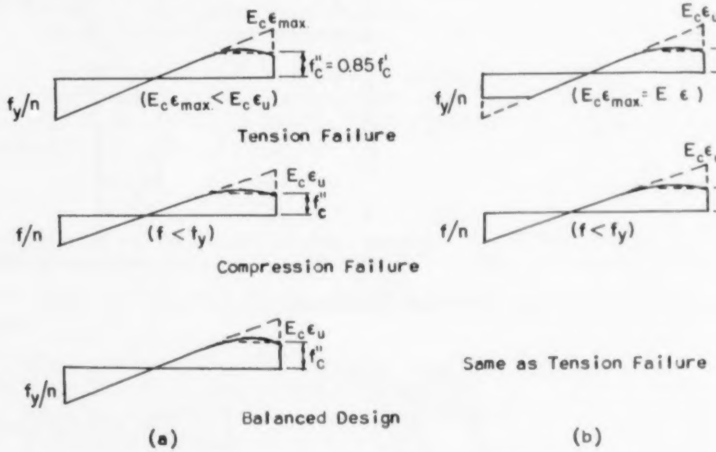
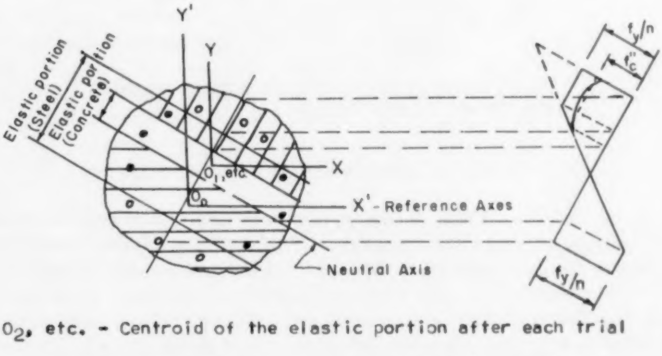


Fig. 4. Illustrations of Various Modes of Failure

(a) Failure is based on the yielding of the extreme steel bar in tension.

(b) Failure is based on the final collapse of member.



O_1, O_2, etc. - Centroid of the elastic portion after each trial

o - Plastic steel bars ● - Elastic steel bars

▨ - Plastic concrete area ▤ - Cracked concrete area

Fig. 5. General Stress Distribution

Basic Principles for Determining the Actual Stress in a Given Section

Let us consider a section subjected to excessive loading which would produce the general stress distribution as shown in Fig. 5.

Let axes x-x and y-y be the centroidal axes for the elastically stressed section. Let P, M_x, M_y be the given loading on the section. Let P', M'_x, M'_y be respectively the axial force and moments sustained by the plastically stressed portion. Let P", M''_x, M''_y be the part of the loading sustained by the elastic region.

Then

$$P'' = P - P' \tag{1a}$$

$$M''_x = M_x - M'_x \tag{1b}$$

$$M''_y = M_y - M'_y \tag{1c}$$

The stresses in the elastic section are given by the following formula,

$$f = \frac{P''}{A_E} + \frac{M''_{oy} - M''_{ox}\dfrac{I_{oxy}}{I_{ox}}}{I_{oy} - \dfrac{I^2_{oxy}}{I_{ox}}}\,x + \frac{M''_{ox} - M''_{oy}\dfrac{I_{oxy}}{I_{ox}}}{I_{ox} - \dfrac{I^2_{oxy}}{I_{oy}}}\,y \tag{2}$$

where subscript "o" is added for the values which should be referred to the centroidal axes of the elastic portion. A_E = area of the elastic portion; I_{ox} = moment of inertia about x axis; I_{oy} = moment of inertia about y axis; I_{oxy} = product of inertia.

The above formula is the familiar formula for sections subjected to unsymmetrical loading within the elastic range.(10)

An article was published in 1940 in Civil Engineering(11) outlining the procedure for analyzing nonhomogenious sections subjected to unsymmetrical loading. To begin the analysis, an arbitrary neutral axis was assumed. The position of the actual neutral axis was found by successive approximations using the above formula. The actual stresses in the member were then determined.

The determination of elastic stresses in a section would also provide a means for determining the strain distribution in a section even if the section is stressed elastoplastically. Since plane sections remain plane after bending, Eq. (2) would give a ficticious elastic stress - f_e at a plastically stressed point. Using this ficticious stress and the modulus of elasticity of the material, it is possible to solve for the strain existing at that particular point.

$$\epsilon = \frac{f_e}{E} \tag{3}$$

and

$$\epsilon_u = \frac{\max f_e}{E} \tag{3a}$$

It is very important to know the value of ϵ as failure will occur only when the maximum strain exceeds the ultimate value $\epsilon_u = 0.003$.

The above considerations furnish a basis for checking the stresses and strains of a given section for both elastic and elastoplastic conditions. It

should be noted that the fully plastic case does not occur in reinforced concrete sections since concrete is unable to develop strains large enough to cause a fully plastic state in the compression area.

The Problem and Its Solution

A reinforced concrete section is given or tentatively selected. The loading (P, M_x, M_y) is given as the design working loads multiplied by the load factors. The load factors are safety factors as the section should be so proportioned that its ultimate strength is equal to or higher than the loading under service conditions multiplied by these factors.

The problem is to find the real stress-strain distribution, i.e., to determine whether the given section has sufficient ultimate capacity. The procedure for solving the problem is as follows:

1. Assume the section is elastic and assume a neutral axis location.
2. Locate the centroid and compute I_x, I_y and I_{xy}.
3. Using Eq. (2), determine the maximum stresses in the concrete and reinforcing. Should elastic limit stresses be exceeded, proceed as in step (5) below.
4. If the maximum steel stress is within the elastic range and the maximum steel stress is within the elastic range and the maximum concrete stress corresponds to a strain less than ϵ_o, solve the problem elastically as follows:

   (a) Using Eq. (2) set $f = 0$ and determine a new neutral axis location. If it does not coincide with the assumed neutral axis, repeat the above steps.

   (b) If in any of the cycles the maximum stresses in steel or concrete exceed the elastic limit proceed as in step (5) below.

5. Locate the new neutral axis by using Eq. (2) with $f = 0$. Use the stress distribution found to start with the next step or use these results as a guide in assuming a new location of the neutral axis and a new stress distribution for the next step.
6. Using Eq. (3) find the boundary of the elastic and plastic regions.
7. Compute the axial load P' and resisting moments M_x' and M_y' sustained by the plastic region and deduct them from the given P, M_x, M_y, respectively. The values obtained are P'', M_y'' and M_x'' which are sustained by the elastic region only.
8. Put the loads P'', M_x'' and M_y'' on the elastic region <u>only</u> and find the maximum stress by repeating steps 2 and 3. Locate the neutral axis by the method shown in step 5. If the new neutral axis coincides with the axis found in step 5, the stress and strain distribution is final.
9. If neutral axes do not coincide, find the stress distribution and strain distribution. Repeat steps 5, 6, 7 and 8. When the actual stress condition is approached, the difference between stresses found in two consecutive trials and also the movement of the neutral axis will become quite small. The procedure as shown above will not only locate the final position of the neutral axis, but will also give the final stress and strain distribution in steel and concrete.
10. If the strain in the concrete would exceed that found in an earlier trial and the neutral axis is moving toward the compressed area, a new

stronger section should be chosen. If the final strain is less than 0.003 then the given loading is below the ultimate strength of the section.

A Numerical Example

In this example, $P = 3{,}000$ kips, $M_x = 12{,}000$ in. kips $M_y = 30{,}000$ in. kips represent the design working loads multiplied by the load factors. The section shown in Fig. 6a was selected and it will be checked for ultimate capacity to support the loading.

The materials were assumed to have the following properties:

$f_y = 40{,}000$ psi $\quad f_c'' = 0.85\, f_c'' = 2{,}550$ psi

$f_c' = 3{,}000$ psi $\quad E_c = 1{,}000\, f_c'' = 3{,}000{,}000$ psi

$\epsilon_u = 0.003$

for which $n = 10$ and $\epsilon_o = \dfrac{0.85\, f_c'}{E_c} = 0.00085$.

Fig. 6b shows the same section with steel bars represented by transformed area strips for simplification of the computation of moments of inertia. In the following computations, concrete areas occupied by steel in the compressive region are not deducted. Although it is possible to adjust either the steel area or the steel stress in that region, such refinement would hardly affect the final results.

Fig. 7 shows the method used in computing the moment of inertia of various sections.

Tables 1 and 2 show the first trial. No neutral axis location was assumed, the entire section being assumed uncracked. The computation of section properties is shown in Table 1 and the computation of stresses is shown in Table 2. Based on Eq. (A) in Table 2, the new neutral axis was located and the maximum stresses were computed. These stresses were higher than the elastic limits in both steel and concrete. The boundaries between the elastic and plastic portions were obtained as shown in Table 2. Fig. 8 shows the locations of these boundaries. The resulting stress distribution is shown pictorially in Fig. 9.

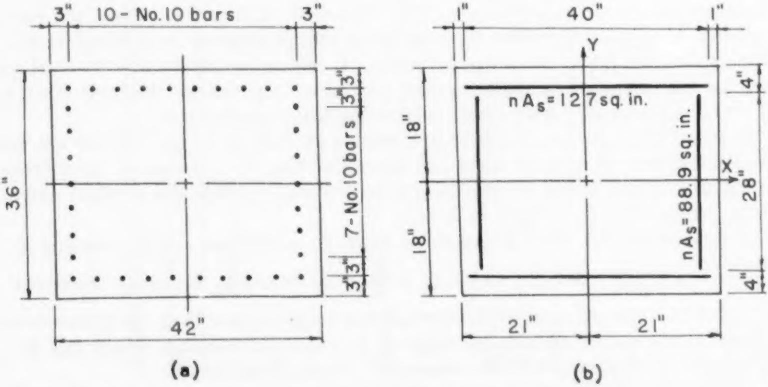
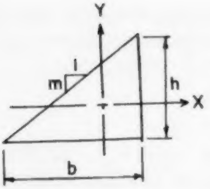


Fig. 6. Rectangular Column Section to be Investigated



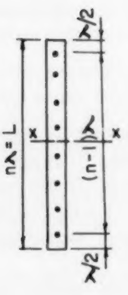
$\bar{I}_y = Ab^2/18$ $\quad A = bh/2$

$\bar{I}_x = Ah^2/18$

$\bar{I}_{xy} = \pm\, Abh/36$ $\quad$ positive when slope m is positive

negative when slope m is negative

Fig. 7a. Moment of Inertia of a Triangular Area



$I_o = a_1\lambda^2 n(n+1)(n-1)/12 = A_s\lambda^2(n^2-1)/12$

where a_1 = area of one bar, A_s = total area of n bars and λ = spacing of the bars

but $(n^2 - 1) \approx n^2$, therefore, approximately $I_o = A_s L^2/12$

where $L = \lambda n$ is the length of an equivalent strip of steel having an area = A

Fig. 7b. Moment of Inertia of a Group of Bars

Table 3 shows the computation of moments of inertia and products of inertia of individual parts of the concrete area for the second trial. Table 4 shows the computation of section properties in detail. Table 5 shows the computation of (P', M'_x, M'_y); (P'', M''_x, M''_y); (P'', M''_{ox}, M''_{oy}), etc., in detail. Table 6 shows stress computations and locations of neutral axis similar to Table 2. Fig. 10 shows the locations of the boundaries between elastic and plastic portions after the second trial.

Tables 7, 8, 9 and 10 are similar to Tables 3, 4, 5 and 6. They are included to show some details when the plastic portions become somewhat more complicated than those shown previously. It should be noted that neutral axis positions and boundaries of the elastic and plastic portions obtained after each trial were used as starting positions for the following trials.

The results of two more trials are shown in Tables 11 and 12 and the successive locations of neutral axes are shown in Fig. 11. It can be seen from Eqs. (D) and (E) in Table 11 and 12 and in the Fig. 11 that the neutral axes in the last two trials practically coincide.

The final stresses were found to be max. f_e = 8.08 ksi corresponding to $\epsilon = \dfrac{\max f_e}{E} = \dfrac{8.08}{3{,}000} = 0.0027$ which is somewhat below $\epsilon_u = 0.003$. However, it is doubtful that the section could sustain loadings higher than the given values. The final maximum tensile f_s is equal to 30 ksi. The section would fail in compression as tensile steel stress is below the yield point.

It should be pointed out that in order to demonstrate the method an unlikely first assumption was made, namely that the entire section was uncracked.

Table 1. Stress Computation - First Trial

Description	A	x	y	Ax	Ay	$Ax^2 + \bar{I}_y$	$Ay^2 + \bar{I}_x$	$Axy + I_{xy}$
nA_s, $n = 10$: 2x10- No.10	254.0	0	$\pm 15^{(1)}$	0	$0^{(1)}$	0 33,900(2)	57,100(1) 0	0 0
2x7- No.10	177.8	$\pm 18^{(1)}$	0	$0^{(1)}$	0	57,600(1) 0	0 11,600(2)	0 0
Total Steel	431.8			0	0	91,500	68,700	0
Concrete 36(42)	1512.0	0	0	0	0	0 222,300(3)	0 163,200(3)	0 0
Total	1943.8			0	0	313,800	231,900	0

Notes:

(1) When 2 areas (each = A_1) are located symmetrically about an axis (say at a distance of $\pm y$ from the x axis), the algebraic sum of their static moment is $A_1 y = 0$. But the moment of inertia is $I_x = \Sigma A_1 y^2 = Ay^2$ where $A = \Sigma A_1 = 2A_1$

$254(15)^2 = 57{,}100$; $177.8(18)^2 = 57{,}600$

(2) $I = AL^2/12$, $254(40)^2/12 = 33{,}900$; $177.8(28)^2/12 = 11{,}600$

(3) $1512.0(42)^2/12 = 222{,}300$; $1{,}512.0(36)^2/12 = 163{,}200$

Table 2. Results of the First Trial

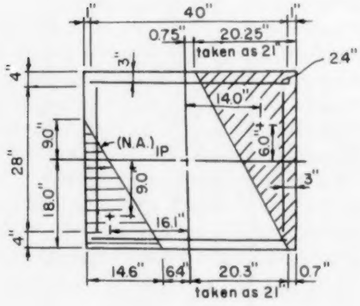
$P = 3{,}000$ kips; $M_y = 30{,}000$ in. kips; $M_x = 12{,}000$ in. kips

$$f = \frac{P}{A} + \frac{M_y}{I_y}x + \frac{M_x}{I_x}y = \frac{3{,}000}{1943.8} + \frac{30{,}000}{313{,}800}x + \frac{12{,}000}{231{,}900}y$$

$$f = 1.543 + 0.0957x + 0.0519y \;\ldots\ldots\; \text{(A)}$$

Computations based on Eq. (A)	x	y	f
Intercepts of N.A. with the boundaries of the section	-21.0	+ 9.00*	0
	- 6.36*	-18.0	0
Concrete stress at the extreme corner	+21.0	-18.0	4.49* > f''_c
Boundaries of the elastic portion for $f''_c = 0.85f'_c = 2.55$ ksi	+ 0.75*	+18.0	2.55 = f''_c
	+20.3*	-18.0	2.55 = f''_c
Steel stress at the extreme corner bars	+18.0	+15.0	40.32* (1)
	-18.0	-15.0	9.56* (1)
Boundaries of the elastic portion for $f_y = 40$ ksi.	+17.6*	+15.0	40.0 = f_y (1)
	+18.0	+14.2*	40.0 = f_y (1)

Note: * Indicates the unknown solved for by Eq. (A).
(1) $f_s = n(1.543 + 0.0957x + 0.0519y)$ with $n = 10$.



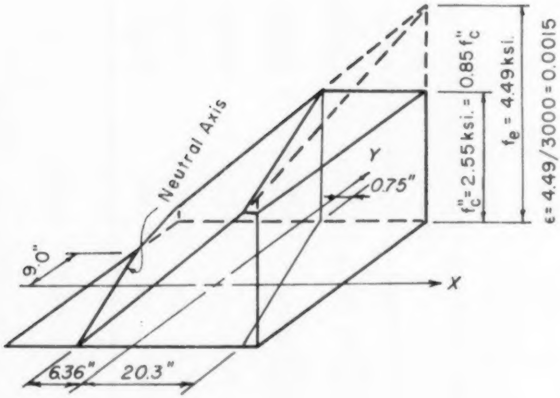
$$A_{SP} = \frac{(2.4)(127)}{(40)(10)} = 0.76 \text{ in}^2$$

Plastic portion of concrete section

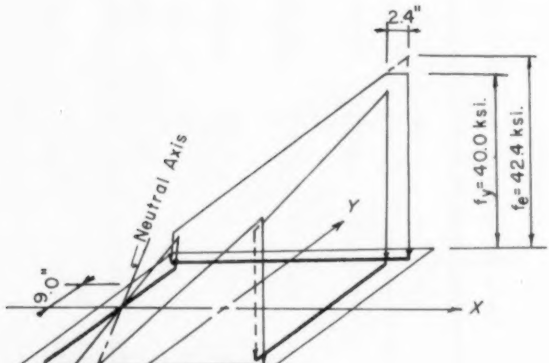
Elastic portion (concrete)

Cracked portion

Fig. 8 Boundary Locations after the First Trial



(a) Concrete Stress



(b) Steel Stress

Fig. 9. Stress Distribution after the First Trial

Table 3. Concrete Section Properties - Second Trial

Description	A	x	y	Ax	Ay	$Ax^2 + \bar{I}_y$	$Ay^2 + \bar{I}_x$	$Axy + \bar{I}_{xy}$
Plastic section $A_{CP} = 21(36)/2$	378	+14	+6	5,300	2,270	74,200 9,250(1)	13,650 27,200(1)	31,800 -7,940(1)
Cracked portion $A_{CC} = 14.6(27.0)/2$	197	-16.1	-9.00	-3,170	-1,775	51,000 2,330(2)	16,000 7,980(2)	28,500 -2,160(2)
$A_{CP} + A_{CC}$	575			2,130	495	136,880	64,830	50,200
Total A_c (Table 1)	1,512			0	0	222,300	163,200	0
$A_{CE} = A_c-(A_{CP}+A_{CC})$	937			-2,130	-495	85,420	98,370	-50,200

Notes:

(1) Based on Fig. 7a
$378(21)^2/18 = 9,250$; $378(36)^2/18 = 27,200$; $-378(21)(36)/36 = -7,940$

(2) Based on Fig. 7a
$197(14.6)^2/18 = 2,330$; $197(27.0)^2/18 = 7,980$; $-197(14.6)(27.0)/36 = -2,160$

Table 4. Properties of Elastic Portion - Second Trial

Description	A	x	y	Ax	Ay	$Ax^2 + \bar{I}_y$	$Ay^2 + \bar{I}_x$	$Axy + \bar{I}_{xy}$
Total steel (from Table 1)	431.8	0	0	0	0	91,500	68,700	0
nA_{SP} (see Fig. 8)	-7.6	+18.8	+15	-143	-114	-2,690 -4 (1)	-1,710 0	-2,140 0
Total steel	424.2			-143	-114	88,806	66,990	-2,140
Concrete (see Table 3)	937			-2,130	-495	85,420	98,370	-50,200
Total	1,361.2			-2,273	-609	174,226	165,360	-52,340
Correction to centroid		-1.67 (2)	-0.447 (2)			-3,796 (3)	-270 (3)	-1,020 (3)
Total I_o						170,430	165,090	-53,360
$-\frac{I_{xy}^2}{I_x}$ or $-\frac{I_{xy}^2}{I_{oy}}$						-17,230 (4)	-16,690 (4)	
Total	1,361.2					153,200	148,400	
	A					$I_{oy} - \frac{I_{oxy}^2}{I_{ox}}$	$I_{ox} - \frac{I_{oxy}^2}{I_{oy}}$	

Notes for Table 4:

(1) $7.6(2.4)^2/12 = 4$

(2) $\bar{x} = \frac{\Sigma Ax}{\Sigma A} = \frac{-2273}{1361.2} = -1.67$ $\qquad \bar{y} = \frac{\Sigma Ay}{\Sigma A} = \frac{609}{1361.2} = 0.447$

(3) $-(\Sigma A)\bar{x}^2 = -\bar{x}\,\Sigma Ax = -(1.67)(-2273) = -3796$
$-(\Sigma A)\bar{y}^2 = -\bar{y}\,\Sigma Ay = -(0.447)(-609) = -270$
$-(\Sigma A)\overline{xy} = -\bar{x}\,\Sigma Ay = -(-1.67)(-609) = -1020$

(4) $-\frac{(53,360)^2}{165,090} = -17,230;$ $\qquad -\frac{(53,360)^2}{170,430} = -16,690$

Table 5. Stress Computation - Second Trial

A	f	P	x	M_y	y	M_x
$A_{SP} = 0.76$ (Fig. 8)	40.0	30	+13.8	570	+15.0	450
$A_{CP} = 378^{(1)}$	2.55	964	+14.0	13,500	+ 6.0	5,780
P', M'_y, M'_x (Total)		994		14,070		6,230
P, M_y, M_x (given)		3,000		30,000		12,000
P'', M''_y, M''_x (2)		2,006		15,930		5,770
Correction to centroid		2,006	$-1.67^{(3)}$	$3,350^{(4)}$	$-0.447^{(3)}$	$896^{(4)}$
M''_{oy}; M''_{ox} (Total)				19,280		6,666
$-M''_{ox}\frac{I_{oxy}}{I_{ox}}$; $-M''_{oy}\frac{I_{oxy}}{I_{oy}}$				$2,150^{(5)}$		$6,024^{(5)}$
Total		2,006		21,430		12,690
		P''		$M''_{oy} - M''_{ox}\frac{I_{oxy}}{I_{ox}}$		$M''_{ox} - M''_{oy}\frac{I_{oxy}}{I_{oy}}$

Notes for Table 5:

(1) Table 3 and Fig. 8

(2) $P'' = P-P'$, $M''_y = M_y - M'_y$, $M''_x = M_x - M'_x$

(3) From Table 4

(4) $-P\bar{x} = -2{,}006(-1.67) = 3{,}350$ $\quad -P\bar{y} = -2{,}006(-0.447) = 896$

(5) $-M_{ox} \dfrac{I_{oxy}}{I_{ox}} = -6{,}666 \dfrac{(-53{,}360)}{165{,}090} = 2{,}150$

$-M_{oy} \dfrac{I_{oxy}}{I_{ox}} = -19{,}280 \dfrac{(-53{,}360)}{170{,}430} = 6{,}024$

Table 6. Results of the Second Trial

$$f = \frac{P}{A} + \frac{M''_{oy} - M''_{ox}\dfrac{I_{oxy}}{I_{ox}}}{I_{oy} - \dfrac{(I_{oxy})^2}{I_{ox}}}x + \frac{M''_{ox} - M''_{oy}\dfrac{I_{oxy}}{I_{oy}}}{I_{ox} - \dfrac{(I_{oxy})^2}{I_{ox}}}y = \frac{2{,}006}{1361.2} + \frac{21{,}430}{153{,}200}x + \frac{12{,}690}{148{,}400}y$$

$f = 1.472 + 0.140x + 0.0855y$ ..... (B)

Computations Based on Eq. (B)	$x^{(1)}$	$y^{(1)}$	f
Intercepts of N.A. with the boundaries of the section	-19.33 + 0.20*	+14.45* -17.55*	0 0
Concrete stress at the extreme corner	+22.67	+18.45	6.23* > f''_c
Boundaries of the elastic portion for f''_c = 2.55 ksi	- 3.57* +18.40*	+18.45 -17.55	2.55 = f''_c 2.55 = f''_c
Steel stress at the extreme corner bars	+19.67 -16.33	+15.45 -14.55	55.44* > f_y (2) -22.57* < f_y (2)
Boundaries of the elastic portion for f_y = 40 ksi.	+ 8.62* +19.67	+15.45 - 2.60*	40.0 = f_y (2) 40.0 = f_y (2)

Note: (1) These coordinates are referred to centroid of the elastic portion. For location of coordinates, see Fig. 10.

(2) $f_s = (1.472 + 0.140x + 0.0855y)n$ with $n = 10$.

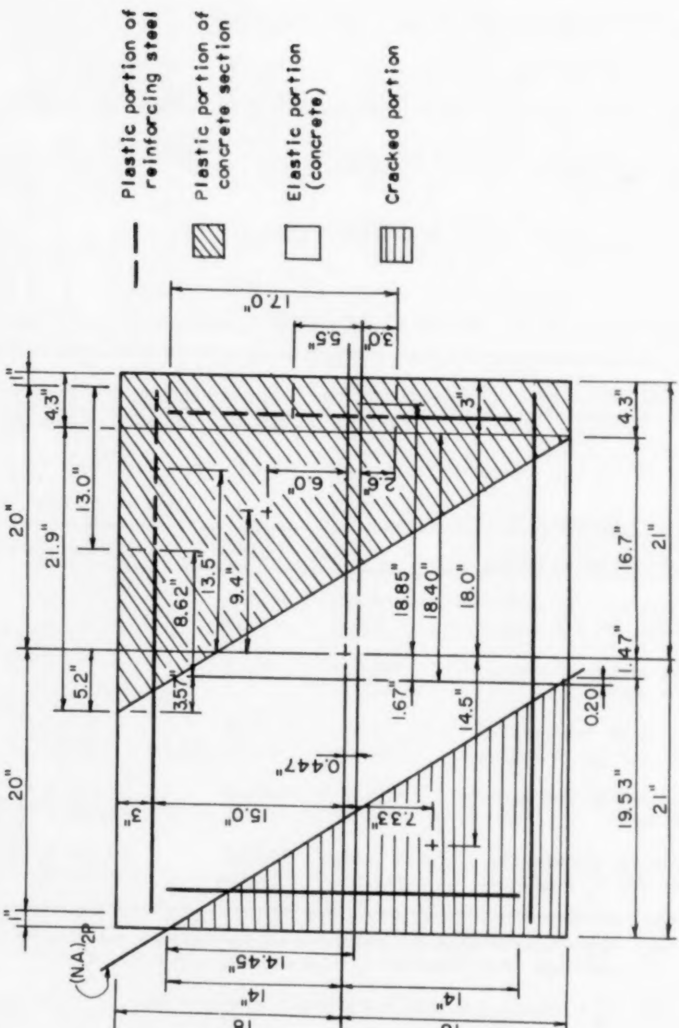


Fig. 10. Boundary Locations After the Second Trial

Table 7. Concrete Properties - Third Trial

Description	A	x	y	Ax	Ay	$Ax^2 + \bar{I}_y$	$Ay + \bar{I}_x$	$Axy + \bar{I}_{xy}$
Plastic section $A_{CP1} = 4.3(36)$	155	18.85	0	2,920	0	55,000 240(1)	0 16,750(1)	0 0
$A_{CP2} = 21.9(36)/2$	394	9.4	6.0	3,700	2,360	34,800 10,500(2)	14,180 28,400(2)	22,200 -8,640(2)
Cracked portion $A_{CC} = 19.5(32)/2$	312	-14.5	-7.33	-4,520	-2,290	65,500 6,600(3)	16,800 17,800(3)	33,200 -5,400(3)
$A_{CP} + A_{CC}$	861			2,100	70	172,640	93,930	41,360
Total A_c (Table 1)	1,512			0	0	222,300	163,200	0
$A_{CE} = A_c - (A_{CP} + A_{CC})$	651			2,100	-70	49,660	69,270	-41,360

Notes:

(1) $155(4.3)^2/12 = 240$; $155(36)^2/12 = 16,750$

(2) $394(21.9)^2/18 = 10,500$; $394(36)^2/18 = 28,400$; $-394(21.9)(36)/36 = 8,640$

(3) $312(19.5)^2/18 = 6,600$; $312(32)^2/18 = 17,800$; $-312(19.5)(32)/36 = 5,400$

Table 8. Properties of Elastic Portion - Third Trial

Description	A	x	y	Ax	Ay	$Ax^2 + \bar{I}_y$	$Ay^2 + \bar{I}_x$	$Axy + \bar{I}_{xy}$
nA_{SP1} = 13.0(12.7)(10)/40 (see Fig. 10)	41.3	+13.5	+15.0	556	620	7,520 580(1)	9,300 0	8,350 0
nA_{SP2} = 17.0(8.89)(10)/28 (see Fig. 10)	54.0	+18.0	+ 5.5	974	297	17,510 0	1,630 1,300(1)	5,350 0
Total A_{SP}	95.3			1,530	917	25,610	12,230	13,780
Total steel (Table 1)	431.8	0	0	0	0	91,500	68,700	0
Total steel - A_{SP}	336.5			-1,530	-917	65,890	56,470	-13,780
Concrete (Table 7)	651			-2,100	- 70	49,660	69,270	-41,360
Total	987.5			-3,630	-987	115,550	125,740	-55,060
Correction to centroid		- 3.68	- 1.00			-13,350	-990	- 3,630
Total (I_o)						102,200	124,750	58,690
$-\frac{I_{oxy}^2}{I_{ox}}$ or $-\frac{I_{oxy}^2}{I_{oy}}$						-27,600	-33,650	
Total	987.5					74,600	91,100	
	A					$I_{oy} - \frac{I_{oxy}^2}{I_{ox}}$	$I_{ox} - \frac{I_{oxy}^2}{I_{oy}}$	

Note: (1) $41.3(13)^2/12 = 580$; $54.0(17)^2/12 = 1,300$

Table 9. Stress Computation - Third Trial

A	f	P	x	M_y	y	M_x
$A_{SP1} = 4.13$ (Table 8)	40.0	165	13.5	2,230	15.0	2,480
$A_{SP2} = 5.40$	40.0	216	18.0	3,890	5.5	1,190
$A_{CP1} = 155$ (Table 7)	2.55	395	18.85	7,450	0	0
$A_{CP2} = 394$	2.55	1006	9.4	9,450	6.0	6,040
P', M'_y, M'_x (Total)		1782		23,020		9,710
P, M_y, M_x (given)		3000		30,000		12,000
P'', M''_y, M''_x		1218		6,980		2,290
Correction to centroid		1218	-3.68(1)	4,480	-1.00(1)	1,218
M''_{oy}; M''_{ox} (Total)				11,460		3,508
$-M''_{ox}\frac{I_{oxy}}{I_{ox}}$; $-M''_{oy}\frac{I_{oxy}}{I_{oy}}$				1,650		6,570
Total		1218		13,110		10,078
		P		$M''_{oy} - M''_{ox}\frac{I_{oxy}}{I_{ox}}$		$M''_{ox} - M''_{oy}\frac{I_{oxy}}{I_{oy}}$

Note: (1) From Table 8.

Table 10. Results of the Third Trial

$$f = \frac{1,218}{987.5} + \frac{13,110}{74,600}x + \frac{10,078}{91,100}y = 1.232 + 0.1760x + 0.1104y$$

Computations Based on Eq. (C)	x	y	f
Intercepts of N.A. with the boundaries of the section	-17.32 + 3.68*	+16.42* -17.00	0 0
Concrete stress at the extreme corner	+ 24.68	+19.00	7.67* $> f_c''$
Boundaries of the elastic portion for $f_c'' = 2.55$ ksi	- 4.42* +18.20*	+19.00 -17.00	2.55 = f_c'' 2.55 = f_c''
Steel stress at the extreme corner bars	+21.68 -14.32	+16.00 -14.00	68.2* $> f_y$ -28.4* $< f_y$
Boundaries of the elastic portion for $f_y = 40$ ksi.	+ 5.68* +21.68	+16.00 - 9.50*	40.0 = f_y 40.0 = f_y

Using the same procedure as before, the following results were obtained:

Table 11. Results of the Fourth Trial

$\overline{x} = +4.98$
$\overline{y} = -1.28$

$$f = 1.01 + 0.1830x + 0.1157y \qquad (D)$$

Computations Based on Eq. (D)	x	y	f
Intercepts of N.A. with the boundaries of the section	-16.02 + 5.03*	+16.60* -16.72	0 0
Concrete stress at the extreme corner	+25.98	+19.28	7.99* $> f_c''$
Boundaries of the elastic portion for $f_c'' = 0.85 f_c'' = 2.55$ ksi	- 4.07* +19.40*	+19.28 -16.72	2.55 = f_c'' 2.55 = f_c''
Steel stress at the extreme corner bars, $f_s = nf$ in Eq. (D)	+22.98 -13.02	+16.28 -13.72	71.0* $> f_y$ -29.5* $< f_y$
Boundaries of the elastic portion for $f_y = 40$ ksi	+ 6.06* +22.98	+16.48 -10.47	40.0 = f_y 40.0 = f_y

Again using the same procedure, the following results were obtained.

Table 12. Results of the Fifth Trial

$\bar{x} = 5.40$
$\bar{y} = -1.37$ $\quad f = 0.929 + 0.185x + 0.117y \quad$ (E)

Computations Based on Eq. (E)	x	y	f
Intercepts of N.A. with the boundaries	-15.60	+16.70*	0
of the section	+ 5.50*	-16.63	0
Concrete stress at the extreme corner	+26.40	+19.37	8.08* $> f_c''$
Boundaries of the elastic portion	- 3.48*	+19.37	2.55 = f_c''
for $f_c'' = 0.85 f_c' = 2.55$ ksi	+19.25*	-16.63	2.55 = f_c''
Steel stress at the extreme corner	+23.40	+16.37	73.5* $> f_y$
bars $f_c = nf$ in Eq. (E)	-12.60	-13.63	30.0* $< f_y$
Boundaries of the elastic portion	+ 5.65*	+16.37	40.0 = f_y
for $f_y = 40$ ksi	+23.40	-10.75*	40.0 = f_y

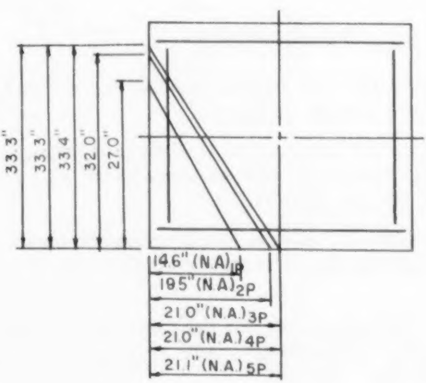


Fig. 11. Successive Locations of Neutral Axis After Each Trial

Also, after determining the first location of the neutral axis and the elasto-plastic boundaries, the movement of the neutral axis to its final position progressed continually. After obtaining some experience, and with a careful examination of the loading, the location of the neutral axis can be guessed fairly accurately. Then, one or two trials will yield the final stress distribution.

CONCLUDING REMARKS

The general method as outlined in this paper can be extended to any sections of irregular shape subjected to any kind of loading. It could be extended to general types of stress-strain curves consisting of an elastic portion and a non-elastic portion. The main advantage of this method is that both ultimate stress and strain are obtained. Thus, the ultimate capacity of the section can be determined analytically with a fair degree of accuracy. Also, the method would be applicable to determine stress-strain distributions for sections carrying loads above or below the ultimate strength.

The elastic solution of unsymmetrical bending problems has been worked out using electronic computers.(12) The procedure suggested in this paper is very similar to the elastic solution and it is hoped that this method can be programmed for electronic computers in the near future.

ACKNOWLEDGEMENT

This paper is based on a thesis submitted in partial fulfillment of the requirements for the degree of Master of Science in the Civil Engineering Department of the Illinois Institute of Technology.

The authors would like to express their indebtedness to Professor E. I. Fiesenheiser M. ASCE for his valuable suggestions and corrections of the original manuscript.

BIBLIOGRAPHY

1. Building Code Requirements for Reinforced Concrete, (ACI 318-56). ACI Journal May 1956, Proceedings V. 52.

2. "Report of ASCE-ACI Joint Committee on Ultimate Strength Design", Proceedings, ASCE V. 81, Paper No. 809, October 1955; ACI Journal, January 1956, Proceedings V. 52, pp. 505-524.

3. Whitney, C. S. and Cohen, E., "Guide for Ultimate Strength Design of Reinforced Concrete". ACI Journal, November 1956, V. 28, No. 5. Proceedings V. 53.

4. Au, Tung, "Ultimate Strength Design of Rectangular Concrete Members Subjected to Unsymmetrical Bending". ACI Journal February 1958, V. 29, No. 8; Proceedings V. 54.

5. Hognestad, E., "A Study of Combined Bending and Axial Load in Reinforced Concrete Members", Bulletin No. 399, University of Illinois Engineering Experiment Station, Urbana, November 1951, 128 pp.

6. Hognestad, E., "Confirmation of Inelastic Stress Distribution in Concrete", Proceedings ASCE, V. 83, Paper 1189, March 1957.

7. Whitney, C. S., "Plastic Theory of Reinforced Concrete Design", Transactions, ASCE V. 107, 1942 p. 251; Proceedings, December 1940

8. Jensen, V. P., "Ultimate Strength of Reinforced Concrete Beams as Related to the Plasticity Ratio of Concrete", Bulletin No. 345, University of Illinois Engineering Experiment Station, Urbana, June 1943, 60 pp.

9. Hognestad, E., Hanson, N. W. and McHenry, D., "Concrete Stress Distribution in Ultimate Strength Design", ACI Journal, December 1955, Proceedings V. 52, pp. 455-479.

10. Cross, Hardy, "The Column Analogy", Bulletin No. 215, University of Illinois Engineering Experiment Station, Urbana.

11. Saville, W. S., "Analysing Non-Homogeneous Sections Subjected to Bending and Direct Stress", Civil Engineering V. 10, March 1940.

12. Kozak, John J. and Shields, Robert E., "Electronic Computers Speed Up Bridge Design". Civil Engineering, V. 27 April 1957.

Journal of the
STRUCTURAL DIVISION
Proceedings of the American Society of Civil Engineers

INSPECTION AND TESTS OF WELDING OF HIGHWAY BRIDGES[a]

John L. Beaton,[1] A.M. ASCE
(Proc. Paper 1866)

SYNOPSIS

This paper discusses the inspection techniques developed by the California Division of Highways for the control of the fabrication of welded highway bridges. Included is the management of such inspection as well as the step-by-step procedure followed. The use of radiography and other forms of nondestructive testing is outlined with special emphasis on the standards used to interpret the radiographic film.

INTRODUCTION

During the past ten years the California Division of Highways has constructed 250 bridges containing about 120,000 tons of welded structural steel, and during the present fiscal year will have had about 60 bridges under various stages of contract which will require about 50,000 additional tons of welded structural steel. These bridges have been designed and the construction supervised by the Bridge Department of the California Division of Highways with the fabrication inspection of such structures being performed by personnel of the Materials and Research Department.

With the possible exception of two steps, the fabrication inspection procedure followed by the California Division of Highways in controlling welded fabrication is approximately the same as that in general use throughout the United States. The two steps which may represent differences from some agencies are (1) a prefabrication conference and (2) the procedure used for radiographic inspection. Both of these steps are described in detail later.

Note: Discussion open until May 1, 1959. To extend the closing date one month, a written request must be filed with the Executive Secretary, ASCE. Paper 1866 is part of the copyrighted Journal of the Structural Division, Proceedings of the American Society of Civil Engineers, Vol. 84, No. ST 8, December, 1958.

a. Presented at the ASCE Convention, June 1958, in Portland, Oregon.
1. Supervising Highway Engr., California Division of Highways.

California has two large industrial centers; the Los Angeles and San Francisco Bay areas. The Materials and Research Department of the Division of Highways supervises an inspection office in each of these areas. It is the duty of these offices to inspect the manufacture of all products destined for highway projects—this includes welded highway structures.

In addition to the normal activities of supervision and coordination, the following items specifically needed by the inspector during the inspection of welding are the responsibility of the Sacramento Headquarters office. These are: (1) to provide design and shop plans and special specifications for each structure, (2) arrange for mill inspection of steel, (3) provide radiographic inspection either by arranging for the services of a local commercial laboratory or furnishing such service through a State-owned mobile radiographic unit, (4) provide machine shop and physical, chemical, and metallurgical laboratory testing services, (5) provide work standards and inspection standards, (6) develop and provide new tests and standards not otherwise covered in published form, and (7) provide welding training and consultation and advice for the more difficult problems. This requires the services of a welding engineer or welding expert.

The documents furnished to the inspector are the American Welding Society Specifications for Welded Highway and Railroad Bridges(1) the American Welding Society Inspection Manual,(2) and Test Method No. Calif. 601.(3) The latter test method has been prepared to cover conditions too new to be included in the A.W.S. publications and those unique to California fabrication and erection. The items of significant difference from usual practice have been included in the Appendix and described below.

In addition to the above, a set of A.S.T.M. Designation E99 Reference Radiographs are made available to each inspector for training and comparison purposes. Fig. 1 in the Appendix shows the standards used by California in interpreting radiographs.

Fig. 2 in the Appendix shows a departure from Appendix B of the A.W.S. Specification D2.0-56.(1) The additional specimens are required as it is considered desirable to test the structural properties of automatic welding as well as the quality of the weld deposit.

Fig. 3, Sheets 1 and 2, shows a departure from the welding operator qualification tests for groove welds shown by Appendix D, Part III, of A.W.S. D2.0-56. These modifications are considered necessary as most of the use of automatic and semi-automatic welding of groove welds in California structures are of the double vee butt joint type.

Fig. 4 shows a departure from the welding operator test for fillet welds shown in Appendix D, Part III, of A.W.S. D2.0-56. This modification is used as being more representative of California's typical application of automatic welding of webs to flanges in bridge girders.

Fig. 5 shows a proposed departure from Appendix D, Part I, of A.W.S. D2.0-56 for use in the qualification of automatic and semi-automatic procedures. This test has not yet been entered into the California specification but has been used on a trial basis for about one year. So far it appears to be a practical and comprehensive test as it provides a measure of the ductility of the fillet as well as the quality. California specifications require a Brinell hardness of the heat-affected zone not to exceed 175. The sectioned test specimen of Fig. 4 is excellent for this determination. The working

PROPOSED CALIFORNIA RADIOGRAPHIC STANDARD

All butt welds designed to carry primary stresses shall be subject to radiographic inspection. The presence of any of the following listed defects in excess of the limits indicated shall result in rejection of the weld as defective:

1. Cracks — no cracking shall be allowed regardless of length or location.

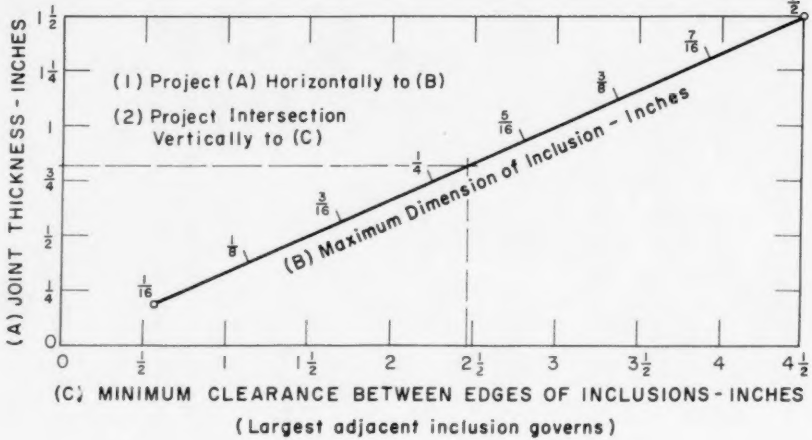
2. Overlaps, lack of penetration, incomplete fusion — none shall be allowed.

3. Inclusions; including slag, porosity, and other deleterious material, — less than 1/16" in greatest dimension shall be allowed if well dispersed such that the sum of the greatest dimensions of the inclusions in any linear inch of welded joint shall not exceed 3/8" and there shall be no inclusion within 1" of the edge of a joint or a point of restraint.

4. Inclusions; including slag, porosity, and other deleterous material, — 1/16" or larger in greatest dimension shall be allowed providing such defects do not exceed the limits indicated by the chart, Sheet 2, of this Standard.

Fig. 1 (Sht. 1 of 2)

SUGGESTED CALIFORNIA RADIOGRAPHIC STANDARDS FOR ALLOWABLE INCLUSIONS



NOTE:

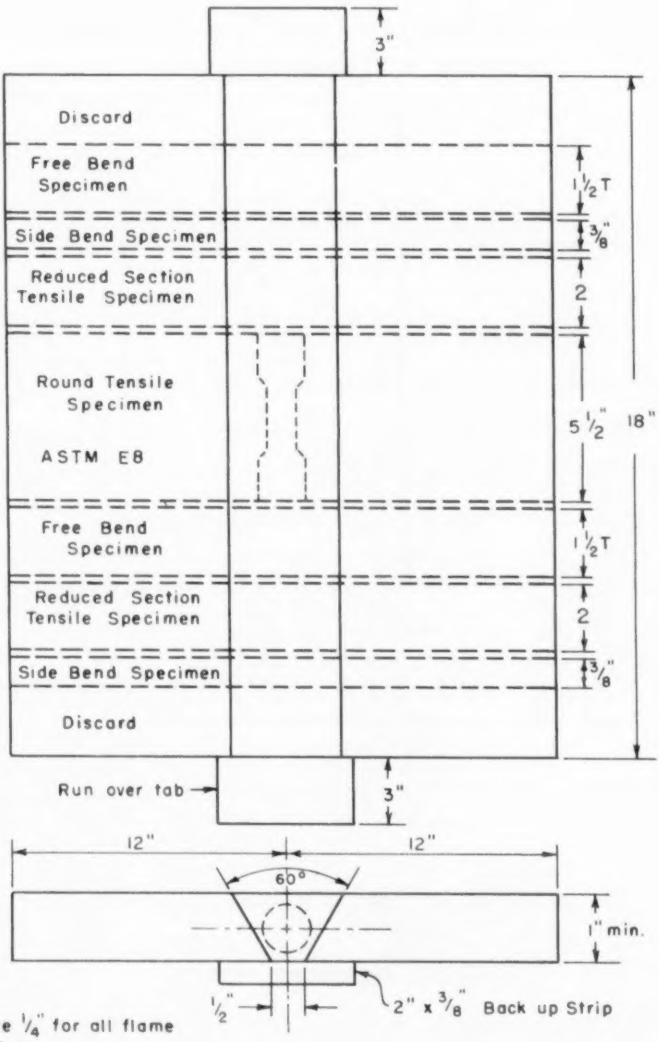
1. Minimum distance from edge of inclusion to free edge of plate or toe of flange fillet weld is twice the clearance between inclusions.
2. Inclusions with any dimension greater than 1/2" are not acceptable.
3. For joint thicknesses greater than 1-1/2", the minimum allowable dimension and spacing of inclusions shall be the same as for 1-1/2" joints.

Fig. 1 (Sht. 2 of 2)

Figure 2

PROCEDURE TEST
Automatic & Semiautomatic
Mechanical Properties of Weld Deposit*

NOTE:
2. Free Bend, Side Bend, and Tensile Specimens shall be prepared and tested as indicated in the A.W.S Standard Specs. Welded Highway & Railway Bridges D2. 0-56, Appendix D.

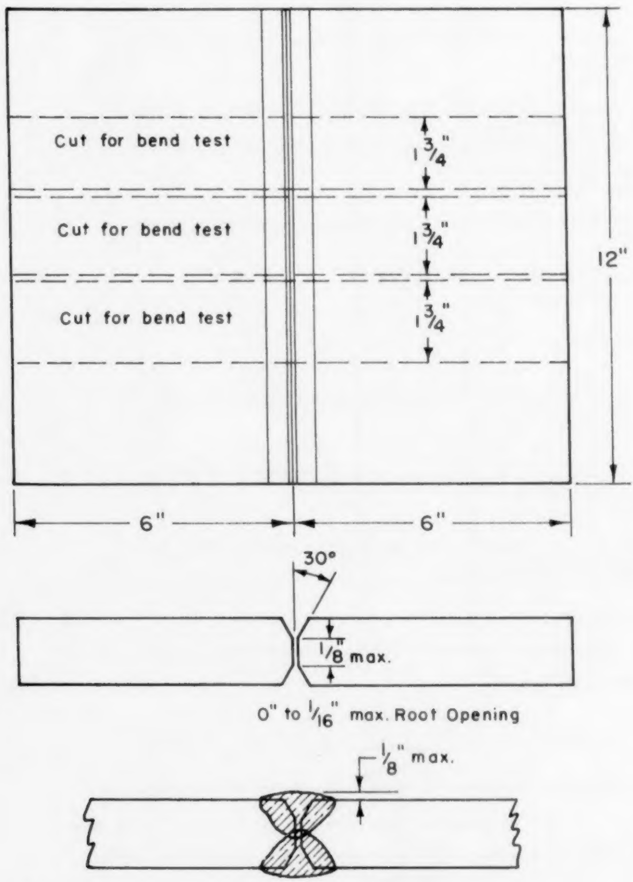


NOTE:
1. Leave 1/4" for all flame Cutting

*Identical test plate to be used for proposed joint qualification, except that proposed joint is to be used rather than as shown and round tensile specimen may be omitted.

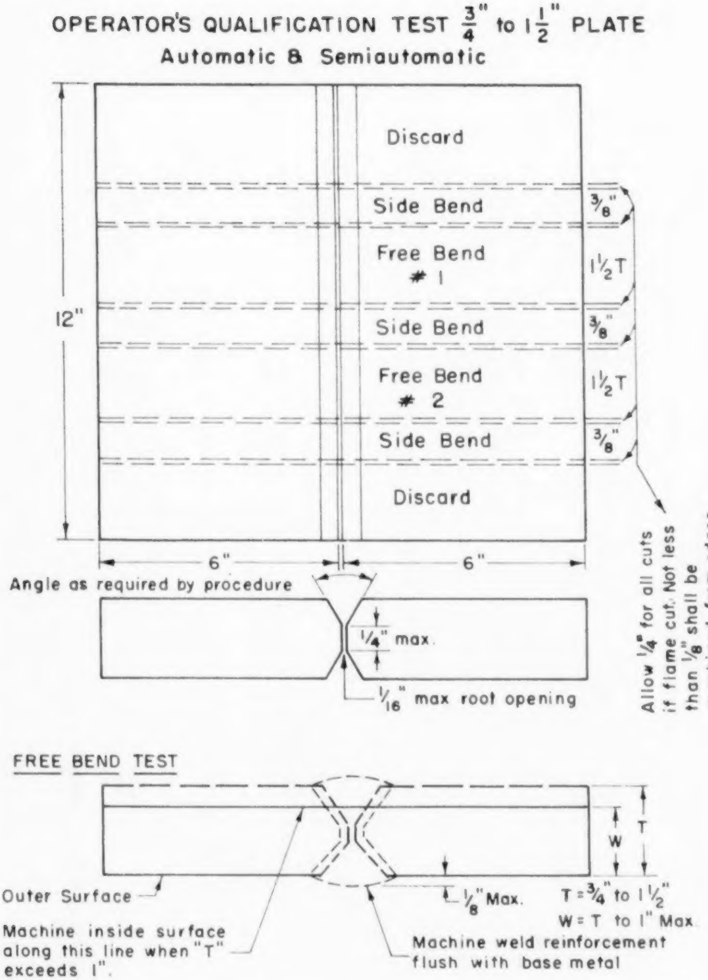
Figure 3
(Sht. 1 of 2)

OPERATOR QUALIFICATION TEST 3/8" to 3/4" PLATE
Automatic & Semiautomatic



Note:
Alternate edge preparation
as required by the procedure
for single & double butt welds.

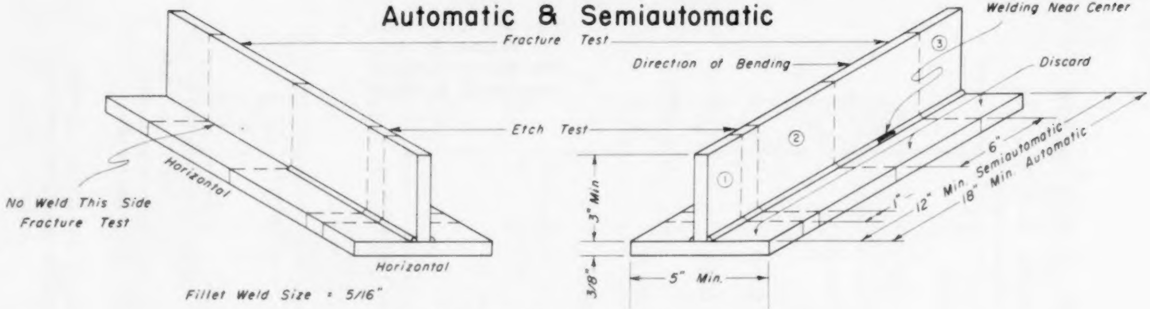
Figure 3
(Sht. 2 of 2)



When making bend tests for plate over 1", place outer surface of weld to outside of bend. The two tests shall be made on opposite sides of the test plates.

Figure 4

OPERATOR & PROCEDURE QUALIFICATION
FILLET WELD TEST
Automatic & Semiautomatic



OPERATOR QUALIFICATION

<u>PREPARATION OF TEST SPECIMENS</u>

Specimens may be sawed, machined, or flame cut from welded test joint. The ends of etched specimen shall be smooth for etching.

<u>TESTING</u>

<u>Fracture Test</u>

The stem of the 6" section shall be loaded laterally in such a way that the root of the weld is in tension. The load shall be steadily increased until the specimen fractures or bends upon itself.

<u>Macro Etch Test</u>

The specimen shall be etched with a suitable solution to give a clear definition of the weld.

<u>RESULTS</u>

<u>Fracture Test</u>

A specimen shall not fracture or if fractured shall not contain defects such as slag, overlap, undercut, etc., totaling more than 1/2". Evidence of cracks in a weld or incomplete root fusion shall constitute grounds for rejection.

<u>Macro Etch Test</u>

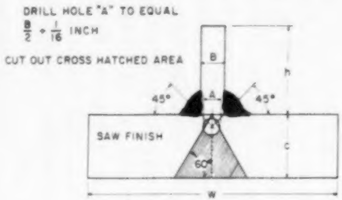
The welds shall show fusion to the root but not necessarily beyond root and be free from cracks. Convexity or concavity of welds shall not exceed 1/16". Both legs of weld shall be equal to within 1/16".

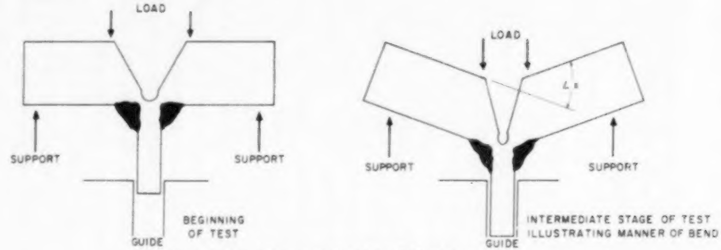
PROCEDURE QUALIFICATION FILLET WELD TEST

Automatic and Semiautomatic

Figure 5

FILLET WELD KEYHOLE "T" BEND TEST





Load and bend to failure or until keyhole "V" notch is closed.
If weld metal or heat affected area fails, weld is not acceptable

SPECIMEN

T BEND SPECIMEN

h = > 4B or 2"
W = > 3C or 5"
Minimum thickness = > $1\frac{1}{4}$ c or $1\frac{3}{4}$"

At least 2 specimen must be prepared from each specimen.

METHOD OF TESTING

I. PREPARATION OF SPECIMEN

Two or more specimen may be sawed or machined (not flame cut) from the #2 portion of the sample as illustrated on fig. 4. These specimen are prepared as shown on fig. 4 and above

II. TESTING

A. Fracture Test
B. Macro Etch Test } See Operator Qualification Test
C. Bend Tests
Specimen shall be loaded and failed as illustrated above.
D. Hardness Tests
Hardness Tests shall be made on a lightly etched section with a suitable machine.

III. TEST RESULTS REQUIRED

A. Fracture Test
B. Macro Etch Test } See Operator Qualification Test
C. Bend Test
∠ x must not be less than 25° at failure
D. Hardness
Brinell Hardness of heat affected zone and parent metal shall be within the following limits

1. Max. Brinell Hardness

$$\frac{\text{Max. Specified or Tested T.S.}^{(b)}\text{ of P.M.}}{500} + 50$$

2. Min. Brinell Hardness

$$\frac{\text{Min. Specified T.S. of P.M.}}{500}$$

This test may be performed with a Rockwell Machine and converted to Brinell Hardness using ASTM conversion chart.

(a) See Sheet 3
(b) Whichever is least
T.S. Tensile Strength
P.M. Parent Metal

procedure to enforce this limit requires that an average of five or more points be used with the average not exceeding 175 with a tolerance for any one point of +5.

Fig. 6 is inspection standards developed by our welding engineer to control the welding of stud shear keys to the tops of steel girders for use in composite construction.

Fig. 7 shows a check list used by our inspection staff as a reminder during the actual inspection process. This list is not carried into the shop as a "check off" list, but rather is used in the office as a reminder.

Welding Inspection Procedure

The technique used during the inspection of the fabrication of California highway bridges can best be illustrated by outlining the procedural details followed in chronological order. Before starting such a discussion, it should be pointed out that the economy of fabrication of a welded bridge is primarily influenced by the designer. In addition to the normal considerations of design, the designer of a welded structure must consider properly balanced geometry, combinations of steel compatible to welding, space considerations and positions for both welding machines and weldors (e.g., tip sizes of electrodes so as to be sure the electrode can be entered into the joint), possible stresses and distortions due to expansion and shrinkage during fabrication as well as possible stresses and distortions during erection. In preparing specifications for welded work, it is of special importance that the engineer brings out points of consideration over and above those that are encountered in the usual fabrication of structural steel. These might include the use of preheat, special restrictions concerning the straightening or preparation of materials or members made of special steels, special electrodes or special equipment required (such as constant potential controllers on welding machines).

The best place for the inspector to initiate fabrication inspection of a welded structure is while it is still on paper on the shop drawings of the fabricator. This is usually accomplished by the fabricator requesting a conference between his staff and the inspector during which time the methods and procedures to be followed by the shop are outlined and the requirements and procedures of inspection are also discussed so that all points of conflict are ironed out in advance. Occasionally the fabricator might indicate that problems of design or erection may be brought up at this meeting, in which case the inspector requests both design and construction engineers to be present. Such a system can only work in those areas where the constructing agency maintains a continuous competent inspection staff. Such conferences give the inspector an opportunity to adjust his procedure to fit into the methods and schedule to be followed by the shop. The inspector's responsibility, during such a conference, to the fabricator is to point out any adverse experience that may have been had with the proposed methods and to call attention to any details that might not be in compliance with the specifications. It is especially important that the inspector discuss any inspection requirements that may delay fabrication, if improperly scheduled, such as welding operator and procedure qualification tests and radiographic inspection.

The first shop contact by the inspector is the inspection of materials before fabrication. The fabricators are instructed to enter on their orders to the mill the fact that the steel is subject to inspection and test witnessing by

Figure 6

INSPECTION OF STUD WELDING FOR COMPOSITE DECK BRIDGE GIRDERS

The following items outline the inspection procedures to govern the work for welding of studs for shear lugs;

1. The process and weldor must be qualified by a procedure test. This test shall consist of welding three studs of the size to be used on a plate of the thickness to be used and then bending the stud at least 30° out of line. The bending is to be done by hammering.

   The welded stud shall indicate complete fusion and exhibit a weld flash or "fillet" for a minimum of 90% of the circumference. There shall be no indication of lack of fusion, or undercut weld. Where the "fillet" does not completely "ring" the stud, fusion shall be clearly defined between the stud and plate.

2. The area where the stud is to be attached must be completely free of all foreign material such as oil, grease, paint, etc. If the mill scale is sufficiently thick to cause difficulty in obtaining proper welds, <u>the scale shall be removed from the weld area by grinding</u>. To do this, it will normally be necessary to grind the area to a bright appearance.

3. After welding, ten (10) studs shall be selected at random along the top of the girder. These 10 studs shall be hammered towards the center of the span out of line 30°. Not more than one of these test studs shall show any signs of failure.

   If more than one stud fails, then all studs in the girder shall be hammered (but not necessarily bent the full 30°) and all that fail shall be replaced. Before replacing the stud, the area shall be ground free of any metal left from the old weld, or in the case of a pocket, it shall be filled with E-6016 weld metal and ground flush.

4. Visual inspection of the completed stud shall show weld flashing not less than 90% around the stud and shall indicate complete fusion. Deficiencies may be corrected by a manual weld of not less than a 1/8" all around fillet of E-6016 electrode.

Figure 7
Sheet 1

CHECK LIST FOR INSPECTION OF STRUCTURAL WELDING

Welding inspection should be performed in sequence with the shop fabrication operations so a minimum of interference between inspection and fabrication will be realized.

A. Prior to Welding

(a) Condition of Base Metal

Visual
Mechanical Properties
Chemical Analysis

(b) Base Metal Defects

Laminations, pipes
Cracks
Surface Irregularities
Flatness

(c) Joint fit up

Edge Preparation
Beveling
Root: Opening and Beveling
Runoff Tabs
Cleaning (Grinding) & (Blast Cleaning)
Backings
Tacking
Dimensions

(d) Assembly Fabrication Set-up

The special set-ups used for assembly fabrication, welding jigs, clamping, aligning, and precambering, etc. (only to see that uniform practices are employed).

B. During Welding

(a) Preheat and interpass temperatures.
(b) Root preparation prior to welding.
(c) Root pass weld.
(d) Second side root preparation (back gouging or chipping).
(e) Cleaning between passes.
(f) Appearance of weld passes. (Comparison with workmanship standards).
(g) Variations from approved welding procedures.
(h) Repair of defective welds.

Figure 7
Sheet 2

C. Acceptance Inspection

(a) Cleaning for inspection.
(b) Nondestructive testing.
(c) Visual inspection. (Surface appearance of welds.) (Conformity of welds with drawings.)
(d) Penetrant dye inspection. (Analysis surface defects.)
(e) Radiography inspection. (Analysis internal defects.)
(f) Proof testing. Structural defects.

D. Repair

(a) Marking for acceptance.
(b) Marking for rejection.
(c) Inspection after repairs.

The inspector will be responsible for the marking of the welding that requires repair, and the acceptance of the repaired welds.

representatives of the State; thus, while the metal is still in the steel mill, it receives a small amount of surface inspection and is identified to mill test reports by a direct representative of the State. The specifications for all California welded work require chemical as well as physical tests of all parent material, except in certain cases where the use of A.S.T.M. Designation: A7 steel is specified. The reports are forwarded to the Sacramento Headquarters of the Materials and Research Department where they are reviewed by the welding engineer before being sent to the assigned inspector. Thus, any special metallurgical problems that might arise during normal welding procedure are called to the attention of the inspector so that he is forewarned. This has been found to be especially important for A.S.T.M. Designation: A242 steels, since the specifications for these steels, as they are now written, permit a wide latitude insofar as the chemical compositions are concerned.

As soon as the steel begins to arrive in the shop, the inspector identifies it to his mill test reports, identifying the steel by heat number. At this time, if it is planned to use several types of steels that might be cut up and so lose their identity, the inspector requests that the shop properly control the various steels so that they can be identified throughout fabrication. One method suggested is to completely spray one side of each plate with a cheap paint so that visual identification is possible throughout fabrication no matter how small the pieces; another method is to spray the diagonal corners of plates and ends of bars and rods. As soon as the shop begins to handle each of the individual slabs of steel, the inspector gives them their first "four side" inspection so as to determine the possible presence of surface defects. (This probably should be done at the mill, but the expense to do so is claimed to be prohibitive.)

The welding electrodes, wire, and flux are thoroughly identified at this time also. This normally is performed by inspection of electrode and flux containers and mill test reports. If there is any doubt in the inspector's mind, such materials are sampled and sent to the laboratory for analysis.

The next step is the qualification of welding procedures and both manual and automatic welding operators. The fabricator should initiate this step well in advance of the actual work so as to minimize delays that may result due to test failures. The general specifications used to control the fabrication of welded highway bridges in California are the Standard Specifications for Welded Highway and Railway Bridges of the American Welding Society, D2.0-56. These specifications prequalify certain joints when made on material conforming to A.S.T.M. Designation: A373 (or in certain cases to A.S.T.M. Designation: A7). Joints other than those listed and all joints in any material other than the two listed above must be prequalified by tests before starting fabrication. This latter fact should be covered in the special project specifications. All procedures and all welding operators must be qualified before the actual welding can commence. Each weldor or welding machine operator must be qualified by test for the position or positions for which he will be used during the conduct of the work. Certificates from other agencies are not recognized; however, if a man has been qualified on previous State work, the inspector may accept that fact as prequalification.

It has been our experience during this qualification period that occasionally the cart is put before the horse and an attempt will be made by the fabricator to qualify a procedure before the operator is qualified. This often results in the procedure being disqualified, not necessarily because there is anything wrong with the procedure, but merely because the weldor could not weld properly.

It should also be pointed out here that, while theoretically the joints prequalified by specification should be joints that can be fabricated with little difficulty by any qualified weldor, it has been our experience that certain of the prequalified groove joints can lead to difficulty if not made to close tolerances, especially where radiographic quality work is required. It is California practice to radiograph each qualification test plate for groove welds before proceeding to the machining. This is relatively inexpensive to do and often eliminates needless machining and testing expense.

With a few exceptions, California follows the current A.W.S. procedure and operator qualification tests exactly. One exception is the use of radiography as discussed above. The balance of the exceptions concern modifications of qualification tests of procedures and operators of automatic and semi-automatic machines. These were discussed previously and are as shown on the Figs. 2 through 5 in the Appendix. The actual steps followed during the manufacture of the various procedure test plates is to manufacture the exact type of joint proposed using the parent metal, electrode (or wire and flux combination), welding machine characteristics, technique, and speed that are to be used during the actual fabrication. For groove type joint, the plate is first radiographed (usually x-ray) and, if it passes, is machined into the various test specimens and the routine tests performed. The same procedure is followed for operator qualification plates except here the joints must conform to the test procedure specifications. The fabricator is offered the option either to machine and test the specimens under the supervision of the State inspector in his plant or to send them to the Sacramento laboratory for machining and testing.

Most of the time the fabricator elects to do the operator qualification tests in his plant so as to expedite the process.

As soon as the material is properly identified and the operators and procedures qualified, then actual fabrication is ready to begin. Actual fabrication is controlled almost entirely by continuous visual inspection. Experienced and competent visual inspection is the key to sound welded structures. An inspector, to be competent in this work, must know and understand welding, must know and understand the specifications and standards of workmanship, and must recognize and know the reasons for the defects that can occur during welding. Several of these defects and their causes have been noted by illustrations on Figs. 11 through 13 in the Appendix.

It must always be recognized that the welding operator, during the instant of welding, is the one man in control of the entire process. Most of the duties of the inspector are finished when the actual welding starts. In other words, if he has made sure that the weldor is qualified, that the joint to be welded is shaped properly and is clean and completely free of moisture, and that the established procedure is being followed, then he need only check for mistakes and equipment failures or variations. If this fact is fully comprehenced, then it is easy to understand why no operator except one highly qualified and completely conscientious in his work should be allowed to work on a major bridge structure.

Occasionally, difficulties will develop during structural welding that are quite complex and require the services of a highly skilled welding engineer or expert to give advice to the welding inspector. The California Division of Highways maintains a metallurgical laboratory in Sacramento to perform routine review of all welding inspection as well as to furnish special advisory service.

Most of the difficulties that occur during the welding of structures, however, are usually traceable to a relatively simple departure from the established procedures, or excessive variations in the mechanical or electrical equipment. Our experience has shown that if the inspectors assure themselves of the following points it is very seldom that difficulties will be encountered. Good practice requires that all joints be clean before welding. This usually requires that the surfaces to be contacted and for at least an inch on each side of the weld area the metal be cleaned so as to be free of all scale, slag, oil, dirt, and other miscellaneous foreign materials. Grinding is the normal cleaning procedure followed; however, there are some shops that use specialized grit blasting machines to perform this cleaning operation. Good practice requires that steel be free of moisture and be reasonably warm so as to be properly welded. Most fabricators find that the use of a preheat of about 300° F. insures a trouble-free joint in that such heat will drive off excess moisture and at the same time will minimize any tendency towards distortion or cracking caused by shrinkage. Good practice requires that once a procedure is established there shall be no deviation from that procedure. Specifications carefully outline the tolerances that are allowed from a qualified procedure; however, in spite of this, it is often necessary to remove many feet of weld metal merely because someone has decided that the machine was not going fast enough, so when the inspector is not looking, the voltage and amperage and speed are turned up with the foolish idea of getting maybe 15% more footage per minute. Even if it was not for the fact that the weld would probably have to be taken out and done over, it is doubtful whether or not this 15% extra in speed is a real gain. This is because under any

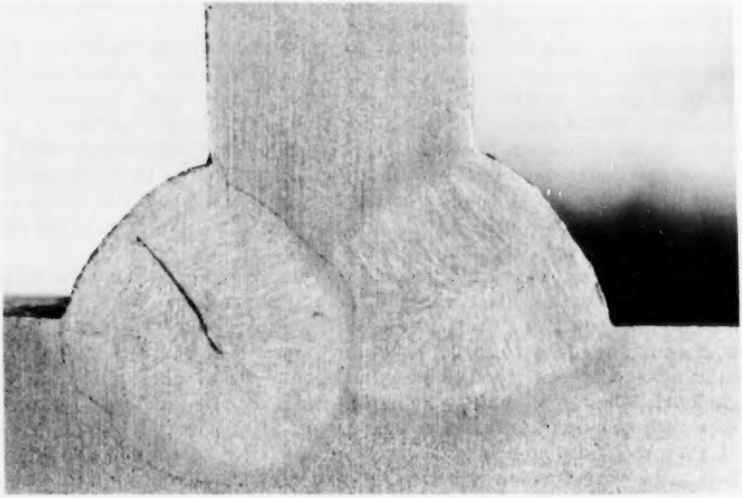


Fig. 11-7. Cracking of this type is associated with submerged arc welding and is the result of attempting to obtain excessive penetration which has resulted in this "familiar" pear shaped weld.

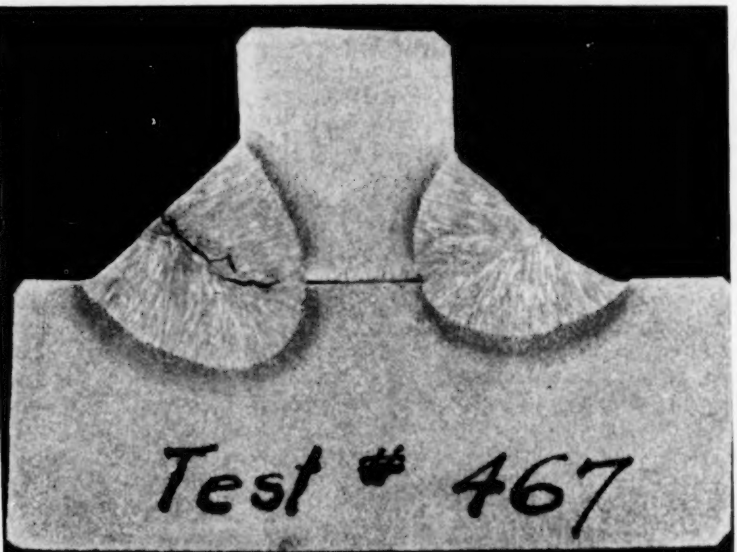


Fig. 11-8. Cracking of this type is also associated with submerged arc welding and is often caused by excessive voltage.

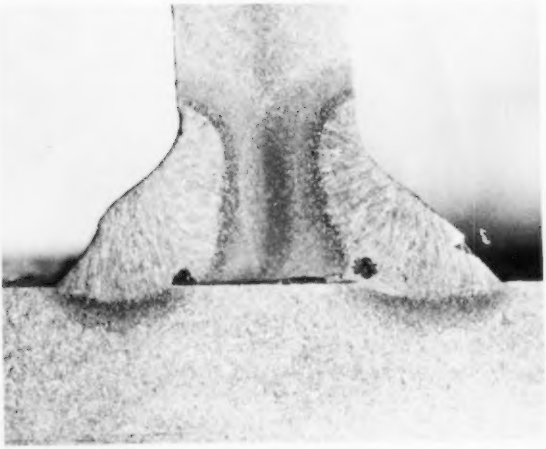


Fig. 12-9. The condition of a fillet weld may be affected by the angle of electrode in relation to the joint to be welded. Here the electrode was allowed to burn off on the weld metal and not in the crater in the parent metal. Lack of fusion and slag in the root of the weld has resulted. This weld was placed using a semi-automatic submerged arc welding machine.

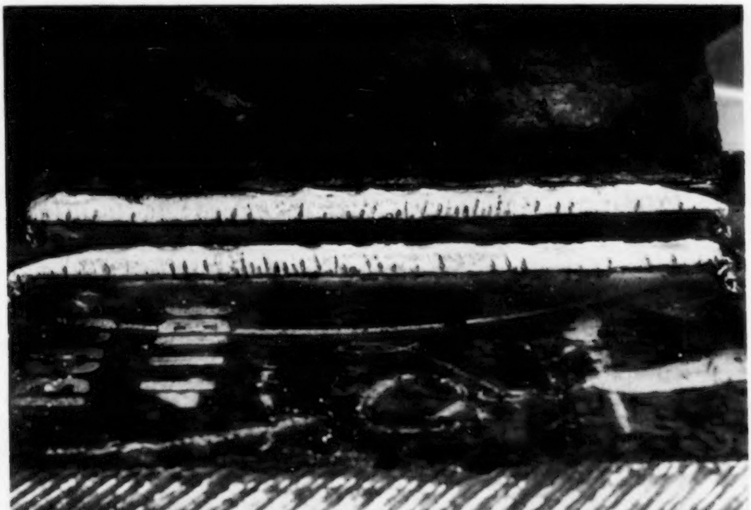


Fig. 12-10. Welding is also affected by moisture, mill scale, rust, oil, dirt, etc. Here can be seen "worm holes" in a fillet weld caused by a combination of the above.

Fig. 13-11. The cleaning of weld splatter, or "dingle berries" as commonly called by the weldor, cannot be ignored. Here can be seen the result of lack of cleaning. As indicated by the arrow, corrosion has started where the weld splatter has dropped off exposing the bare metal.

Fig. 13-12. This shows a section taken from a defective fillet weld, commonly called "roll over". Notice the deep crevice which shows lack of fusion. This is a magnification at the toe of the weld. The naked eye does not reveal this incipient crack, but it always exists in the presence of "roll over" or overlap.

circumstances it takes extremely efficient handling of the steel to get much benefit out of any increase in speed during the actual welding operation. Occasionally, variations beyond the control of the operator can cause defective welding. This can sometimes be detected by changes in the sound of the arc during the actual operation but is usually detected by a post-fabrication inspection by observance of the shape and condition of the weld.

There are certain tools that an inspector finds absolutely necessary during his visual inspection of the completed weld. The first is a penetrant dye kit. This is used to check for surface cracks or by grinding down under the surface to trace a sub-surface crack. Our inspection staff also occasionally makes use of a portable hardness testing machine to check the hardness of the weld and heat affected zone. Such machines, however, are difficult to use in tight places or on sloping surfaces. Special advantage is taken by the inspector of the runover tabs used during the production of all weldments. Such tabs are used by the fabricator to assure that the weld, whether it is a butt joint or a fillet weld, is carried completely full size throughout the entire joint. The tabs are extra pieces of metal tagged onto the base metal at the end of the joint. After the work is completed, they are removed and discarded. The inspector sections, polishes, and etches a sampling of such runover tabs so as to be able to inspect the internal structure of the weld. These can also be sent to the Sacramento laboratory for metallurgical examination. Since it is not required that the parent metal for such tabs conform to specifications, such examination can be considered only indicative. They are not necessarily representative of the structure. Some use is also made of trepanning tools (Fig. 10) and magnetic particle inspection for special investigation into some of the more complex defects that can occur during shop fabrication. In addition there are various miscellaneous small tools such as gages, flashlights, etc. that are useful to the inspector.

Radiography is used on all welded highway structures in this state. It is used as a supplement to visual inspection of welded butt joints. With the exception of special cases, it is never used on fillet welds. Normal procedure calls for 80% of all butt welds designed to resist primary tensile stresses to be radiographed and 25% of all butt welds to carry primary compression stresses to be radiographed. Due to the lack of a nationally accepted radiographic standard for welded highway bridges, this state has developed and is using the radiographic standards shown on Fig. 1 in the Appendix. These standards are thoroughly discussed in a paper presented to the 37th Annual Meeting of the Highway Research Board.(4)

Radiographic work performed in the fabrication shop normally requires the use of one of the radioactive isotopes. Usually Cobalt 60 is used for this purpose, although it may be Cesium 137 or Iridium 192. This requires that an area be roped off for the safety of personnel during the actual radiographic operation. The area to be roped off depends upon the power of the radioactive source. It is the policy of the Division of Highways to conduct this operation so as to interfere as little as possible with the normal operation of the fabrication shop. Since most butt welding is performed before the plates are actually fabricated into the complete member, it is to the shop's advantage that the butt weld be radiographed immediately after completion and before assembly into the completed member. A discussion with shop management, therefore, will usually result in an area being developed in the shop that can be used for such inspection purposes and still not interfere with shop production.

Fig. 10-5. Here a Materials and Research inspector is trepanning a 1/4" core from a fillet weld for examination during a special investigation.

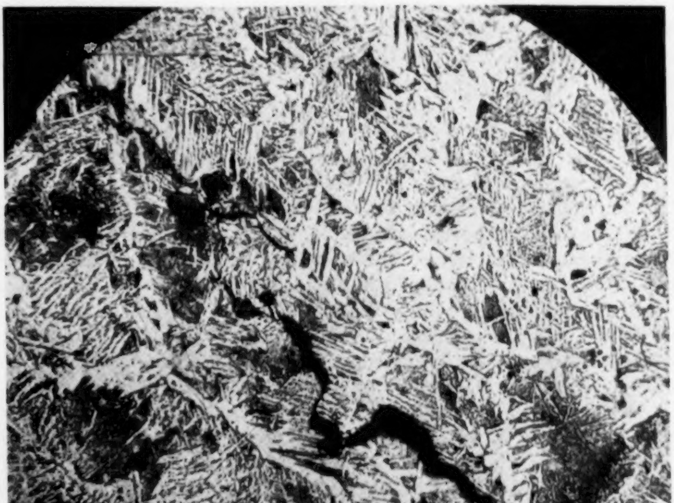


Fig. 10-6. The 1/4" core obtained by trepanning is sectioned and etched for examining for defects and structure such as can be seen here.

The weakest link in the inspection of welding is the lack of a comprehensive nondestructive test of fillet welding. All of the present methods are limited to some degree by various physical or economic factors. Magnetic particle and penetrant dye are limited to surface investigation only. If used below the surface, the weld must be ground or gouged to expose the subsurface, which is impractical for extensive use. Because of the length of most fillet welds, radiography is excessively expensive except for spot checking. Even then it is not entirely reliable because of the varying thicknesses of material through which the exposure usually must be made. Trepanning is classified as a nondestructive test but is of little value in disclosing unknown defects because of the impracticability of trepanning sufficient area to constitute a practical survey. Its value is as a special tool to remove known defective material for metallurgical examination (Fig. 10).

This results in reliance being placed almost entirely on visual inspection, which is sufficient in most cases. Unfortunately, there are occasions when the defect may be beyond the experience of the inspector or cannot be thoroughly observed because of light conditions or other shop imposed limitations. Under these conditions a comprehensive economical nondestructive test would be desirable.

At the present time the most promising solution to this problem appears to be sonic testing. The laboratory of this department is experimenting with this method and, as soon as standards can be developed, will place it in use.

Final inspection is made after the member is sandblasted. This inspection is to determine the preparedness of the surface to receive and hold paint. At this time all the welding has been inspected. However, a final superficial examination is made of the welds so as to locate certain surface defects that are difficult to locate before the blasting operation. These are: (1) trapped slag inclusions in coped corners of stiffeners, (2) excessive splatter, (3) surface condition of steel (scale, scab, surface laminations, etc.), (4) undercutting of welds.

Field Welding

The Bridge Department of the California Division of Highways considers the welding of primary structural members in the field to be a projection of shop fabrication into erection. The Laboratory is therefore requested to supervise the inspection of all primary field welding.

The standards followed in the field are the same as those used during shop fabrication. This means that all weldors and procedures must be qualified before the steel can be joined and that all the before discussed points of good practice must be followed.

The same policy is also followed in the use of radiography, the one difference being that the state-owned mobile radiographic equipment shown on Fig. 8 in the Appendix is usually used in the field whereas the services of a commercial agency are usually used in the shop. The employee assigned to this mobile unit is a highly skilled radiographic technician especially trained in the inspection of welded construction. The mobile unit is equipped so that this inspector can qualify the welding operators, inspect the fabrication and radiograph the joints. Thus he provides a complete welding inspection service to the Resident Engineer.

In general the welding practice followed by the erectors on California

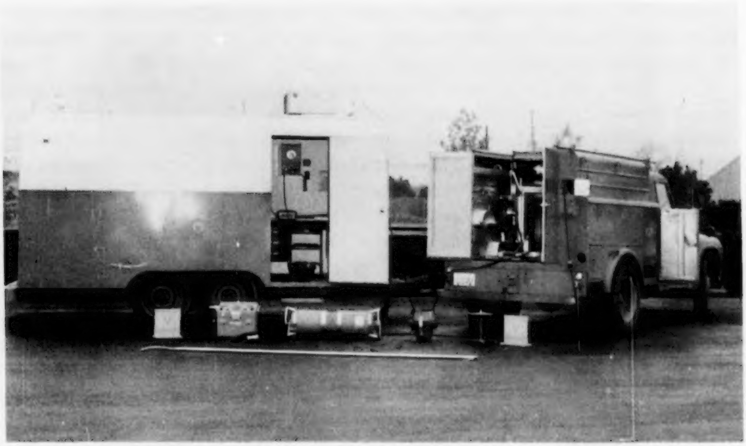


Fig. 8-1. Materials and Research radiographic truck and trailer. This unit is equipped with a portable 175 KV G.E. x-ray machine, 20 curie Cesium 137 isotope in a portable projector, and 1 curie of Cobalt 60 to use as a point source. The unit is equipped with power supply for the x-ray machine and the trailer for darkroom film dryer, lights, film viewers, and power tools, grinders, drills, and hoist mounted in the truck. The radiographic trailer back opens up for a small shop for testing of weldor qualifications.

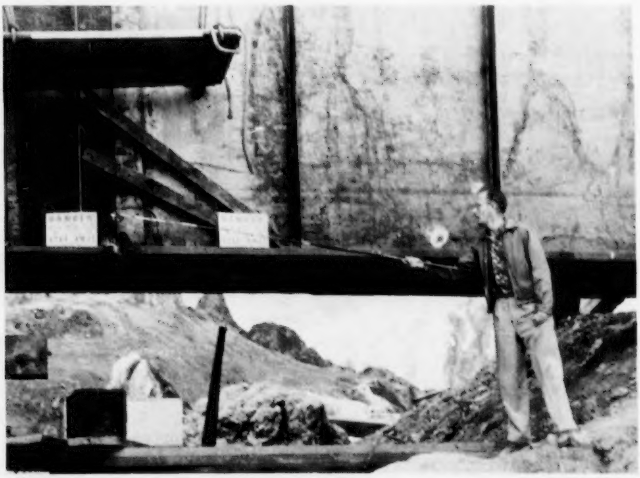


Fig. 8-2. A typical field radiographic operation. Here the radiographic technician is placing the Cobalt 60 source into the container for radiographic exposure on the bottom flange of the bridge girder field splice.

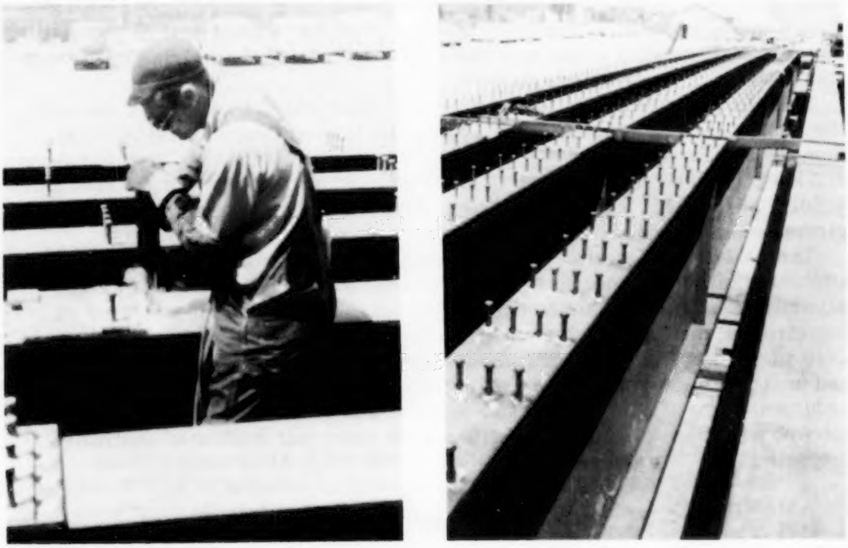


Fig. 9-3. Typical welded stud operation in progress and completed for composite deck action.

Fig. 9-4. A 7/8" test stud welded to a plate and bent by hammering to approximately 30° angle for soundness test.

bridge projects conforms to all other areas subject to good practice controls. There is one procedure, however, that has provoked a good deal of pro and con discussion. This is the procedure followed during the welding of the butt field splices of large bridge girders.

It is the practice of California erectors, encouraged by the State, to weld the flanges first and then the web. Normally the top flange is welded first, then the bottom flange and the web welded last following a back step sequence and keeping both flanges heated to about 300° F. at the start and all during the welding of the web. At the conclusion of the welding of the web, the entire girder is allowed to air cool uniformly.

The reason this practice was adopted and is encouraged rather than that advocated in A.W.S. Specification D2.0-56 Section 504(g) is that our early experience with this latter method was adverse. Each time it was used on our girders, which normally consist of heavy flanges and relatively light but deep webs, this A.W.S. specification suggested procedure resulted in cracking and unsightly distortion of the web. Since switching to the presently used procedure of welding flanges first and web last and preheating the flanges during the web welding, we have had no difficulties under our method of controlled inspection.

REFERENCES

1. Standard Specifications for Welded Highway and Railway Bridges, D2.0-56. American Welding Society, 33 West 39th Street, New York 18, N. Y.
2. Inspection Handbook for Manual Metal-Arc Welding, B1.1-45. A.W.S., 33 West 39th Street, New York 18, N. Y.
3. Materials Manual, Testing and Control Procedures, California State Printing Plant, Documents Section, North 7th Street and Richards Blvd., Sacramento, California.
4. Radiographic Inspection of Welded Highway Bridges, John L. Beaton, 37th Annual Meeting Highway Research Board, Washington, D. C.

Paper 1867 December, 1958 ST 8

Journal of the
STRUCTURAL DIVISION
Proceedings of the American Society of Civil Engineers

DIRECT DESIGN OF OPTIMUM INDETERMINATE TRUSSES

Louis M. Laushey,[1] M. ASCE
(Proc. Paper 1867)

SYNOPSIS

A method is proposed for the direct design of indeterminate trusses. Analysis of trial structures is avoided. Simultaneous equations are not needed for multiply redundant trusses. The direct-design method concentrates on design, not on the mathematical analysis of a trial structure. The principle of potential work is introduced to obtain the maximum compatible stresses for the bars. Redundants are selected to yield the minimum weight of truss. The final bar areas follow directly by dividing the forces in static equilibrium by the stresses satisfying continuity. The relative weights of alternative structures and the optimum structure are revealed by the direct-design method.

INTRODUCTION

The difficulty in designing indeterminate trusses has been stated concisely by Parcel and Moorman.(1) "The stresses in a statically determinate structure depend only on the loading and geometrical configuration and can be computed before the proportioning of the sections. A statically indeterminate structure on the contrary cannot be analyzed except on the basis of a fully proportioned structure, since the stresses are controlled by the elastic distortions and these, in turn, by the distribution of the material. The design of such structures is essentially a 'cut-and-try' process; a structure must be assumed, the redundants determined, the stress calculated, and the parts proportioned; if the section values differ substantially from those originally assumed, the process is repeated — one or more times, as necessary. In

Note: Discussion open until May 1, 1959. To extend the closing date one month, a written request must be filed with the Executive Secretary, ASCE. Paper 1867 is part of the copyrighted Journal of the Structural Division, Proceedings of the American Society of Civil Engineers, Vol. 84, No. ST 8, December, 1958.

1. Thoms Prof. and Head of Dept. of Civ. Eng., Univ. of Cincinnati, Cincinnati, Ohio.

general, there is no assurance that a redundant structure can be designed so that stresses in all its members or parts will correspond to a predetermined, specified value, as is the case for a determinate structure."

The methods to be presented in this paper resemble the straightforward steps in the design of determinate trusses. Stresses of the highest possible working value (balanced stresses) are chosen to satisfy continuity (compatible stresses). Optimum reactions and bar forces are chosen to satisfy statics. The final bar areas are then obtained directly by dividing the forces by the stresses. The calculations usually required to design a multiply redundant structure are greatly reduced and simplified by the proposed method which avoids both the simultaneous equations and the repeated analyses required for a convergence toward an indefinite optimum design.

In contrast, the usual analytical method first requires an assumption of the relative bar areas, then calculations of the redundants and bar forces, and finally determinations of the trial stresses. From these stresses obtained from the first trial, new areas must be assumed and the process repeated until the bar stresses are somewhere near their maximum allowed working values.

Historical Background

The analysis of indeterminate trusses has been presented extensively in the literature, and need not be repeated here. Westergaard's scholarly paper, "One Hundred Fifty Years Advance in Structural Analysis,"(2) cites references to the well-known fundamental laws originated by Maxwell, Mohr, Muller-Breslau, Castigliano, and others. More recent writers(3-11) have extended these principles, but only from the viewpoint of the analysis of some assumed structure. The design of an indeterminate truss, however, remains "cut-and-try," with little progress having been made beyond the usual assumption of equal areas in all bars for the first trial analysis.

Necessary Calculations

A good design of an indeterminate structure requires calculations blended with the art and judgment of the designer. Those calculations that must be made should aim to provide:

1) Compatible Stresses — Bar stresses (really strains) that satisfy the continuity of the structure, that allow no misfits at the joints and no displacements at the reactions;

2) Balanced Stresses — Compatible bar stresses, all equal to the allowable design specifications—or as near as may be, subject to the limitations of continuity; and

3) Optimum Reactions — Redundant forces (and bar forces), in static equilibrium, and of such magnitudes to yield the minimum total weight of truss bars.

A truss with balanced compatible stresses satisfies continuity and economy of material, and coupled with optimum reactions in static equilibrium provides bar areas for the optimum design. These calculable items can be obtained directly to furnish a sound design that can be reconciled easily with any practical considerations.

The Direct-Design Method

A simple example, Fig. 1, will illustrate the direct design method. Reactions for the loads were selected arbitrarily, to satisfy statics, and the bar forces were immediately determined, Fig. 1(a). The optimum reactions will be discussed later, but any set in equilibrium that makes a structure suitable to the designer can be used. Continuity is satisfied with the familiar equation:

$$\delta = \Sigma \frac{FUL}{AE} = \Sigma \frac{SUL}{E} = 0$$

$$\Sigma SUL = 0 \tag{1}$$

which states that the deflection at a redundant is zero. The terms are defined and used later as follows:

F = force in a bar (kips), tension (+), compression (-);
s = stress in a bar (ksi), tension (+), compression (-);
u = signed force in a bar due to one pound in place of a redundant, with no real loads on the structure;
L = length of a bar;
A = area of a bar;
E = modulus of elasticity.

Since the summation of Equation (1) is zero, liberties can be taken to use proportional lengths of the bars, to apply more than one pound as a virtual load if convenient, and to omit the term for the modulus of elasticity when the structure is of one material.

Returning to the example, Fig. 1(b) shows the relative values of u and uL for the bars. Now in principle, the design stresses could be chosen by trial to make Σ suL = 0, and the final areas obtained by dividing the assigned forces by these stresses. But in fact, as will be demonstrated later, ΣuL = 0, and we can write:

$$\Sigma SUL = t\overset{t}{\Sigma} UL + c\overset{c}{\Sigma} UL = 0 \tag{2}$$

where the two summations of uL are made first over the tension bars, then over the compression bars, and where

t = design stress (ksi) in the tension bars, and
c = average design stress (ksi) in the compression bars.

Throughout this paper, variations in allowable compression stresses dependent on particular slenderness ratios will be considered a higher-order of precision to be adjusted during the final choice of sections.

In Fig. 1(c) of this example, each of the two partial summations of uL was zero, which is not surprising when the sum of the two is zero, Equation (2) is satisfied with both t and c being any desired design stress, say 20 ksi and 15 ksi, respectively. Fig. 1(d) shows the final areas obtained directly by dividing the forces of Fig. 1(a) by the compatible stresses of Fig. 1(c). Checks on statics are Σ FL = 0 and ΣuL = 0, and on continuity by ΣsuL = 0; shown at the bottom of Fig. 1.

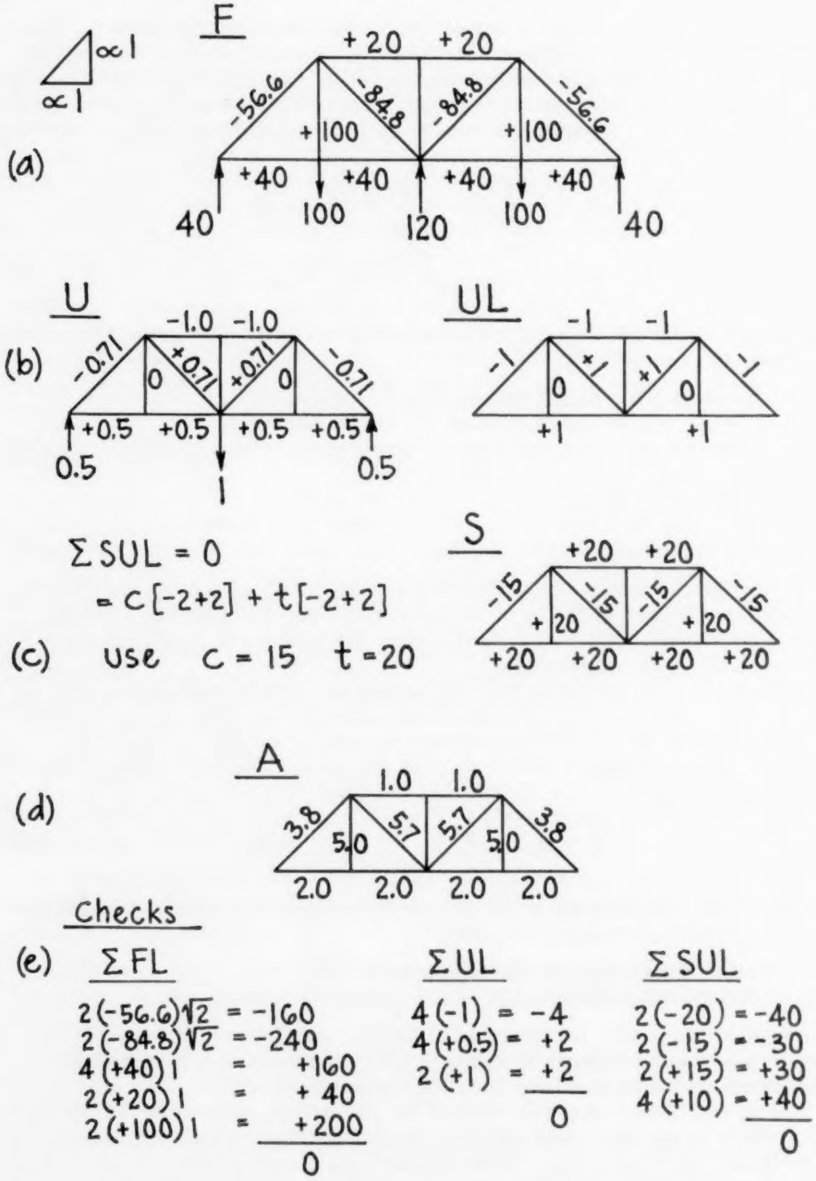


Fig. 1 General Method.

Potential Work

Balanced compatible stresses can be determined with the aid of the principle illustrated by Fig. 2. The truss of Fig. 2(a), loaded on the top chord with one pound, will first be considered. The load of one pound has potential energy with respect to the resultant of the reactions. The resultant reaction is assumed to act along the same line of action as the load, but in the opposite direction. This virtual potential work of $1 \times d$ is defined positive because the tips of the vectors would have to be moved along their lines of action, in the direction of their action, to touch; then there would be no potential energy possessed by the pair of equal and opposite forces. Internally, each bar is loaded at its ends with equal and opposite forces, separated by the length of the bar. Using the same sign convention for the virtual potential work of the bars, compression bars would have positive potential work, and tension bars would have negative potential work.

The hypothesis that the potential work of the loads and reactions equals the potential work of the bars is always proved by test. In Fig. 2(a), there is this virtual potential work:

Externally, for the load and resultant reaction, $1 \times d$, and
Internally, for the bars loaded at their ends, Σ-uL.

Then, in Fig. 2(a), $\Sigma uL + 1 \times d = 0$.

Fig. 2(b) shows that $\Sigma uL = 0$ when the virtual load is placed on the same load line as the reactions. We here interpret the external load and the resultant of the reactions to be collinear vectors with their tips touching, with no virtual potential work. The sum of the virtual potential work of the internal forces, applied at the ends of the bars, then must also be zero; and $\Sigma uL = 0$.

For the real loads of Fig. 2(c), ΣFL is equal to the negative summation of each external force multiplied by its distance from its point of application to a convenient datum through the point of application of the reactions. In Fig. 2(d), $\Sigma FL = 0$ when all loads and reactions (including the redundant) are applied against the datum. Figs. 2(e) and (f) illustrate how the principle can be applied also to internally redundant trusses. In Fig. 2(f), there being no external loads, $\Sigma uL = 0$.

Theorems of Potential Work

For both virtual and real loads: In any truss (determinate or indeterminate) the sum of the product of each bar force and the respective length of the bar must equal the negative sum of the product of each external load and the distance their points of application must be translated to intersect the resultant of the reactions. Or more generally: The potential energy between the points of application of all bar forces must equal the potential energy between the points of application of the external loads and reactions. The principle applies also to non-parallel, non-concurrent force systems.
Symbolically:

$$\Sigma FL + \Sigma(P,R)d = 0 \qquad (3)$$

All reactions and loads are frequently vertical and act on a common horizontal line; then:

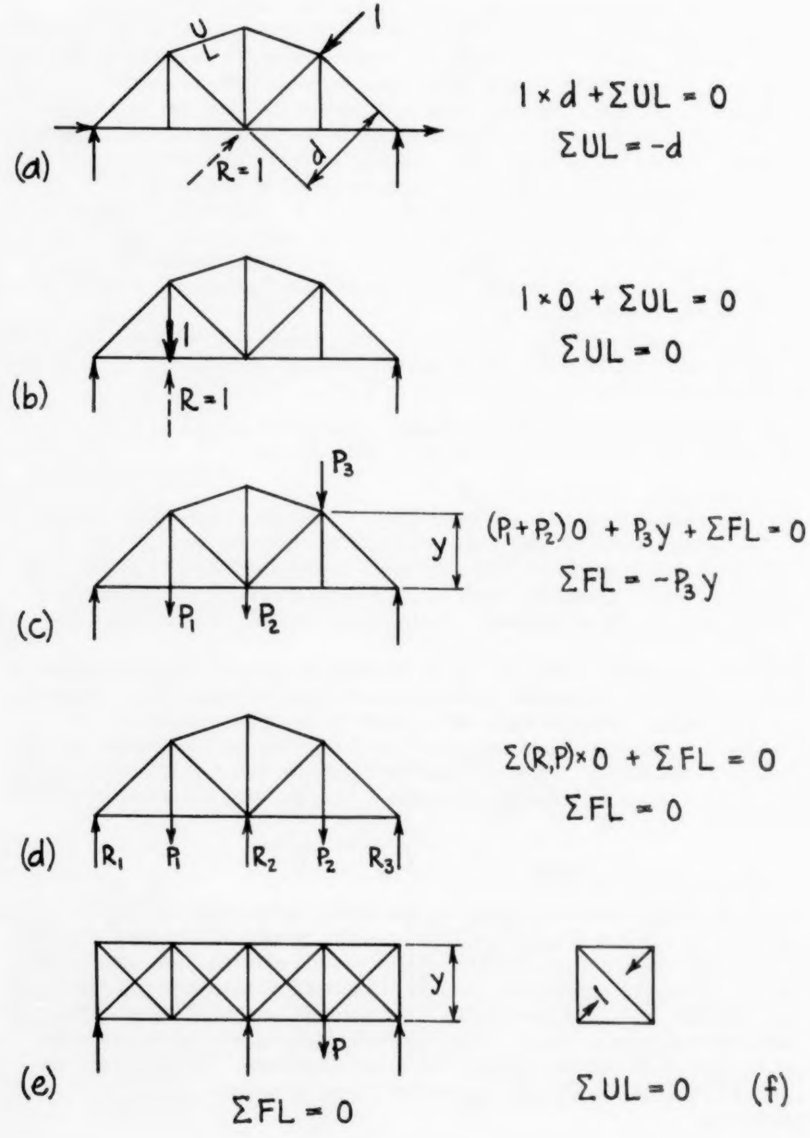


Fig. 2 Potential Work.

$$\Sigma FL = 0 \tag{4}$$

External redundants usually act on a line with the other reactions; then:

$$\Sigma UL = 0 \tag{5}$$

Balanced Compatible Stresses

In summary, the aim is to satisfy the continuity with balanced stresses in this way:

Required - $\Sigma\, suL = c \overset{c}{\Sigma} uL + t \overset{t}{\Sigma} uL = 0$

Since often - $\Sigma\, uL = \overset{c}{\Sigma} ul + \overset{t}{\Sigma} ul = 0$

Try to make - $\overset{c}{\Sigma} uL = \overset{t}{\Sigma} ul = 0$

Then - c and t are free to be any desired values.

But if $\overset{c}{\Sigma} uL$ and $\overset{t}{\Sigma} uL$ are not equal to zero, adjust some values of c and t to satisfy $\Sigma\, suL = 0$.

Check on Statics

Equations (3) and (4) are useful as checks on the statical determination of the bar forces in any trussed structure, whether determinate or indeterminate. The check is rapid, especially when many bars of the same length permit the multiplication of partial sums of forces by constant chord or diagonal lengths, or the cancellation of forces of opposite signs in the chords or in the diagonals. A somewhat less general proof of Equation (3) was obtained from statics by Van den Broek[12] for a check on the calculated bar forces in statically determinate trusses.

In the form of horizontal and vertical components of the bar forces, modifications of Equation (3) are:

$$\Sigma FL = \Sigma F_x L_x + \Sigma F_y L_y \tag{6a}$$

Since $F_x = F \cos \theta$ and $F_y = F \sin \theta$,

and $L_x = L \cos \theta$ and $L_y = L \sin \theta$,

then it is proved that

$$\Sigma F_x L_x + \Sigma F_y L_y = \Sigma FL \cos^2\theta + \Sigma FL \sin^2\theta = \Sigma FL$$

Non-vertical loads and reactions are most conveniently handled as horizontal and vertical components, where:

$$\Sigma(P,R)d = \Sigma(P,R)_x X + \Sigma(P,R)_y Y \tag{6b}$$

Here X and Y are respectively the horizontal and vertical distances the external components have to be translated to touch any rectangular axes perpendicular to the motions. A distance is positive if a force (always positive) is moved in the sense of its action, and negative if the force is moved opposite to the sense of its action.

It also follows that static equilibrium requires that:

$$\Sigma F_x L_x + \Sigma (P,R)_x X = 0 \tag{6c}$$

$$\Sigma F_y L_y + \Sigma (P,R)_y Y = 0 \tag{6d}$$

The theorem of potential work for force components then is: The potential work of the external force components equals the potential work of the internal force components, measured with respect to any two right-angled reference axes. Furthermore, the external and internal potential work must be equal along any axis.

Range of Optimum Reactions

Good designs can be obtained by choosing the arbitrary reactions to be approximately those obtained by moment distribution, assuming a constant moment of inertia. Theoretically, any reactions in equilibrium could be assigned, and the areas obtained to make the structure produce these reactions. Practically, a wild choice of reactions will result in an uneconomical structure; continuity being satisfied only by some bars of low stress. The bar forces would also be larger than necessary to carry the loads, and the bars would be interfering by loading other bars.

Fig. 3(a) shows a two-span truss loaded equally at each panel point. The signs of the bar stresses for reactions obtained by moment distribution are also shown. However, Fig. 3(b) shows that a range of end reaction values from 1.5 P to 2.0 P would retain the same sign of the stress in each bar. The moment distribution value of 1.77 P is almost exactly midway in the range. The calculated values of uL are shown on Fig. 3(c). Once again, the sums of uL equal zero over both the tension and compression bars. Therefore, $\Sigma suL = 0$ with any values of t and c for any of the end reactions between 1.5 P and 2.0 P. This is the range of the optimum reactions, all of which allow the usual design stresses in all of the bars.

The Beam Analogy

Fig. 3(d) illustrates the relation between the direct-design method and the distortions of a singly-indeterminate truss. There are points of inflection at certain panel points (depending on the reaction amounts) where the signs of the stresses change. Between the points of inflection, the bar stresses (really strains) tend to produce an upward deflection at the redundant reaction; the remainder of the structure tends to produce a downward deflection. The upward deflection arises from forces with signs opposite to uL, the downward deflection from forces with signs the same as uL. The two opposing deflections, of course, must cancel. The structure humps between the points of inflection, and sags outside — just like a continuous beam.

The two partial summations of Equation (2) are ideally made zero when: (a) in the tension bars (always associated only with negative values of uL in the hump and only with positive values of the uL in the sag) there are the same positive and negative sums of uL; and (b) in the compression bars (always associated only with positive values of uL in the hump and only with negative values of uL in the sag) there are the same positive and negative sums of uL. These requirements cannot always be satisfied exactly. However,

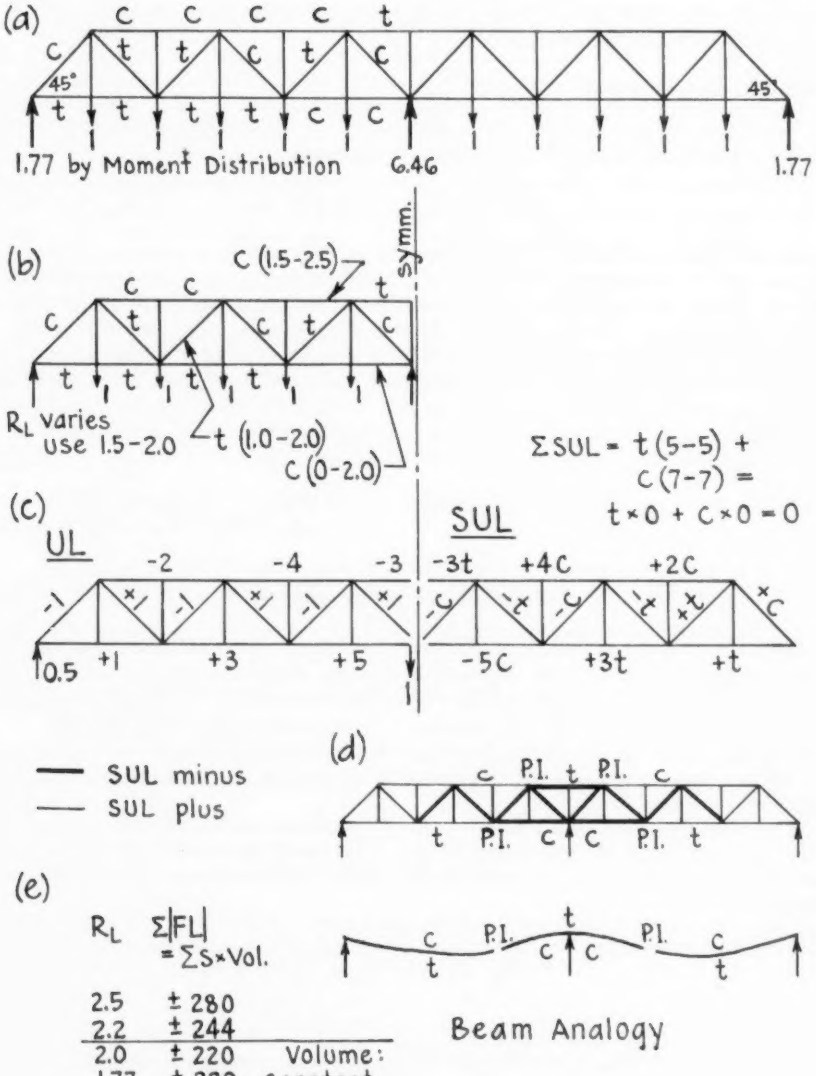


Fig. 3 Optimum Reactions – Minimum Volume.

most structures can be assigned nearly balanced stresses if any of the optimum reactions are assigned.

Comparison of Trusses

The truss weights of alternative structures can be compared and computed by the sum of the values of FL, disregarding signs.

$$\Sigma|FL| = \Sigma|S|AL = \Sigma|S|\,\mathrm{Vol} \tag{7}$$

Since the optimum structures will have the highest possible stresses, the volume will be a minimum when $\Sigma|FL|$ is a minimum. Fig. 3(e) is a tabulation of $\Sigma|FL|$ for a wide range of reactions. One end reaction is of interest in the comparison: the one with 2.5 P, this representing two simple trusses. Assuming the design stresses, which can be chosen here of any amount, would be the same as those used for simple spans, the ratios of $\Sigma|FL|$ will be indicative of the relative metal required. In Fig. 3(e), the bars of all optimum indeterminate trusses require a ratio of 220 to 280 of the metal in two simple spans, or about 79%. The weight of the metal would be:

$$W = \omega\Sigma AL = \omega\Sigma\frac{FL}{S} = \omega\sum^{t}\frac{FL}{t} + \omega\sum^{c}\frac{FL}{c} \tag{8}$$

Minimum Weight Trusses

Optimum trusses result from optimum reactions which yield signs of bar forces in agreement with the signs of the highest allowable compatible stresses. Fig. 3 showed that a range of optimum reaction values produced the same stress sign in any bar, and made zero sums of uL over both the tension and compression bars. Equation (2) showed that when these two sums are zero, t and c could be any desired value – generally the usual working stresses.

Moment distribution often gives a set of reactions in the optimum range. It is important to know whether other reactions would be preferred to minimize the required volume of metal in the truss. The minimum volume will be found by:

$$\frac{\partial}{R_1}(\mathrm{Vol}) = \frac{\partial}{R_1}\Sigma AL = \frac{\partial}{R_1}\Sigma\frac{FL}{s} = \Sigma\frac{u_1L}{s}$$

$$\Sigma\frac{u_1L}{s} = \sum^{t}\frac{u_1L}{t} + \sum^{c}\frac{u_1L}{c}$$

The stresses here are compatible and constant, not dependent on the specific values of the reactions in the range. The derivative of the volume is zero when the two sums of uL are zero. The same volume must then be required for all trusses for which unrestricted constant values of t and c can be assigned. If there is more than one redundant, similar equations (with new sub-scripts) and the same conclusions result.

If the values of uL cannot be made practically zero over both the tension and compression bars, different weights of trusses are required for the various reactions in the range yielding the same stress signs. It is logical that the absolute minimum volume would result from one of the two limits of the optimum range. This set of reactions would form the common boundary between the optimum range and the second-best range adjacent to it. But there is more to a good design than just minimum weight. It is probable that any reaction in the optimum range will yield a good structure. The justification is that the part-sums of uL would be as nearly equal to zero as the geometry allows, and then the stresses would be as high throughout as compatibility allows.

Theorems of Minimum Weight

The principles of virtual potential work have made it possible to determine the minimum weight trusses, whether the redundants are external, internal, or both. The specific principles are:

1) If all sums of uL are zero for both the tension and compression bars, then

a) Compatibility will be satisfied with any constant values of the tension and compression design stresses.

b) Constant minimum weight trusses result from any reactions which produce bar force signs agreeing with the compatible design stresses.

2) If the sums of uL are not zero for both the tension and compression bars, then

a) Compatibility will be satisfied only with unbalanced stresses.

b) Variable weights of trusses result from different reactions, but all reactions which make the sums of uL most nearly equal to zero should be optimum — remembering that other loadings and other requirements must also be considered.

Miscellaneous Corrollaries

It has been proved that all redundants within the optimum band will lead to a minimum, or near minimum, weight of metal. There are similar additional conclusions.

Deflections

The alternative optimum trusses would be designed to attain the same stresses in corresponding members. An application of the principle of angle changes on a conjugate beam leads immediately to the conclusion that these alternative designs have the same angle changes and the same deflection curve.

Work

If the deflection curves are the same, the external loads must move through the same distances. The external work is then the same, and so is the internal work. The theorem of least work by Castigliano, that the redundants

result to make the strain energy a minimum, can then be extended to read: "any of the optimum range of redundants can result to make the strain energy the same minimum."

Volume Change

The change in volume due to the extension and contraction of the bars is:

$$\Delta Vol = \Sigma A \delta L = \Sigma \frac{FL}{E} \tag{9}$$

Thus the change in volume is proportional to ΣFL. The volume is conserved when the loads and reactions act against a common datum, where $\Sigma FL = 0$. The conservation of volume is then consistent with no potential energy between the points of application of the loads and reactions. An interpretation of the potential energy represented by ΣFL follows from Fig. 2(c) where the change in volume is:

$$\Sigma \frac{FL}{E} = \frac{-P_3 y}{E} = \frac{-sAy}{E} = -A\Delta y$$

This shrinkage in volume over the truss is the amount that would occur in one bar of any area interposed directly between the point of application of P_3 on the top chord and the point of application of the resultant of the reactions.

Examples of Usable Designs

Figs. 4 and 5 present the essentials needed for the optimum designs of a series of Warren-type trusses. Panel point loads are equal, representing dead load plus perhaps uniform live load. Suitable modifications can be made for unequal loads.

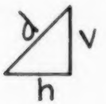
Two-Span Trusses

The optimum range of end reactions is indicated on each sketch of Fig. 4. The range was determined by first obtaining the key value in the range by moment distribution, and then obtaining the signs of the bar forces. A few trials determined the limits that the reactions could have to maintain the stress sign in each bar. The bar forces can be obtained by statics as soon as the reactions are assigned. Since many trusses cannot have balanced stresses throughout, the best compatible stresses can be obtained by an expansion of each of the terms of Equation (2) as follows:

$$t \sum^{t} uL + t' \sum^{t'} uL = 0 \tag{10a}$$

$$c \sum^{c} uL + c' \sum^{c'} uL = 0 \tag{10b}$$

Here t and c are to be the design stresses in the portion of the truss contributing to either positive or negative deflection, and t' and c' are the design stresses in the remainder of the truss. Each sum of uL includes only those items associated with each of the four design stresses. In

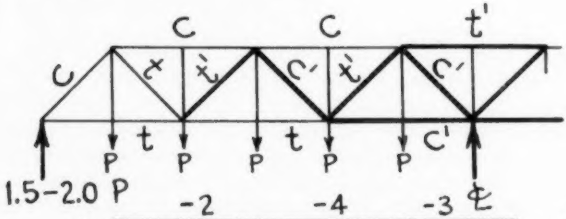


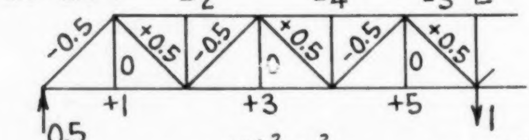
SUL −

SUL +

t, t', c, and c' are design stresses satisfying continuity. Reaction values are optimum range for uniform loads.

Span Ratios 6-6





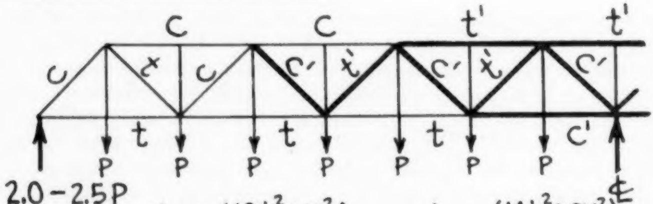
UL

chords, $\times \frac{h^2}{v}$

web, $\times \frac{d^2}{v}$

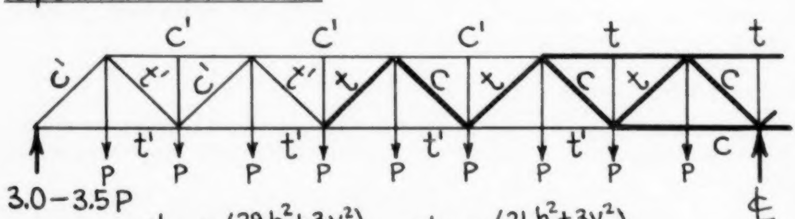
$$t' = t\left(\frac{9h^2+v^2}{8h^2+2v^2}\right) \qquad c' = c\left(\frac{13h^2+v^2}{12h^2+2v^2}\right)$$

Span Ratios 8-8



$$t' = t\left(\frac{19h^2+v^2}{22h^2+2v^2}\right) \qquad c' = c\left(\frac{14h^2+2v^2}{17h^2+3v^2}\right)$$

Span Ratios 10-10



$$t' = t\left(\frac{29h^2+3v^2}{34h^2+2v^2}\right) \qquad c' = c\left(\frac{21h^2+3v^2}{26h^2+2v^2}\right)$$

Fig. 4 Two-Span Models.

Equation (10) and Figs. 4, 5 and 6, t and c would be the usual average design values, and t' and c' would be lower (or equal) values required to satisfy continuity. These stresses, when used with any optimum reactions, yield near minimum weight designs. A more practical option would be to choose all of the reduced stresses in the few bars near the points of inflection where the forces are small, the understresses to be selected by trial to satisfy Σ suL = 0.

Three-Span Trusses

Most trusses with three equal spans can be designed with practically balanced stresses. The equations on Fig. 5 show this is true for all reasonable ratios of panel length to depth. The efficiency of these structures results from the almost complete cancellation of the values of uL over both the tension and compression bars. The procedure involved only one significant difference from the two-span calculations. In Fig. 5, both one-pound loads were applied simultaneously. This was permissible because of the symmetry of both the stresses and uL. It should be remembered that compatibility requires that $\Sigma su_1L = 0$ and $\Sigma su_2L = 0$; or because of the symmetry, $\Sigma s (u_1 + u_2) L = 0$. It would not be necessary that the loads, reactions, or bar forces be symmetrical about the truss center line. It follows then that the compatible stresses of both Figs. 4 and 5 are also usable for non-uniform loads if the signs of the bar forces agree with those given on the sketches. Obvious changes in the stresses can be made to fit severely unequal loadings that shift the points of inflection.

Unequal-Span Trusses

Fig. 6 shows two unequal three-span trusses that would be typical of the many possibilities. The examples are included to indicate how the beam analogies are helpful in pairing the location of the points of inflection with the numerical work, and in visualizing the distortions of the structures. The two structures show that there is an element of luck involved in being able to obtain perfectly balanced stresses. The truss with a short center span can be equally stressed throughout for all load arrangements that are nearly uniform over the structure. The truss with a longer center span is geometrically unable to be equally stressed. It must not be inferred from these two examples that all trusses with short center spans can be equally stressed, or that all trusses with long center spans cannot be equally stressed. Since the sums of uL are composed of items that change magnitude in steps, and change signs only at panel points, the sums cannot always be exactly zero. Nevertheless, the techniques presented illustrate clearly the best design that is possible, and how to complete it.

Curved-Chord Trusses

These same methods could be used for curved-chord trusses, or trusses loaded on the top chord. Further illustrations of specific structures would be redundant.

Design for Live Loads

The problem of handling live loads centers on the influence lines. The usual analytical methods obtain tentative influence lines only after trial areas

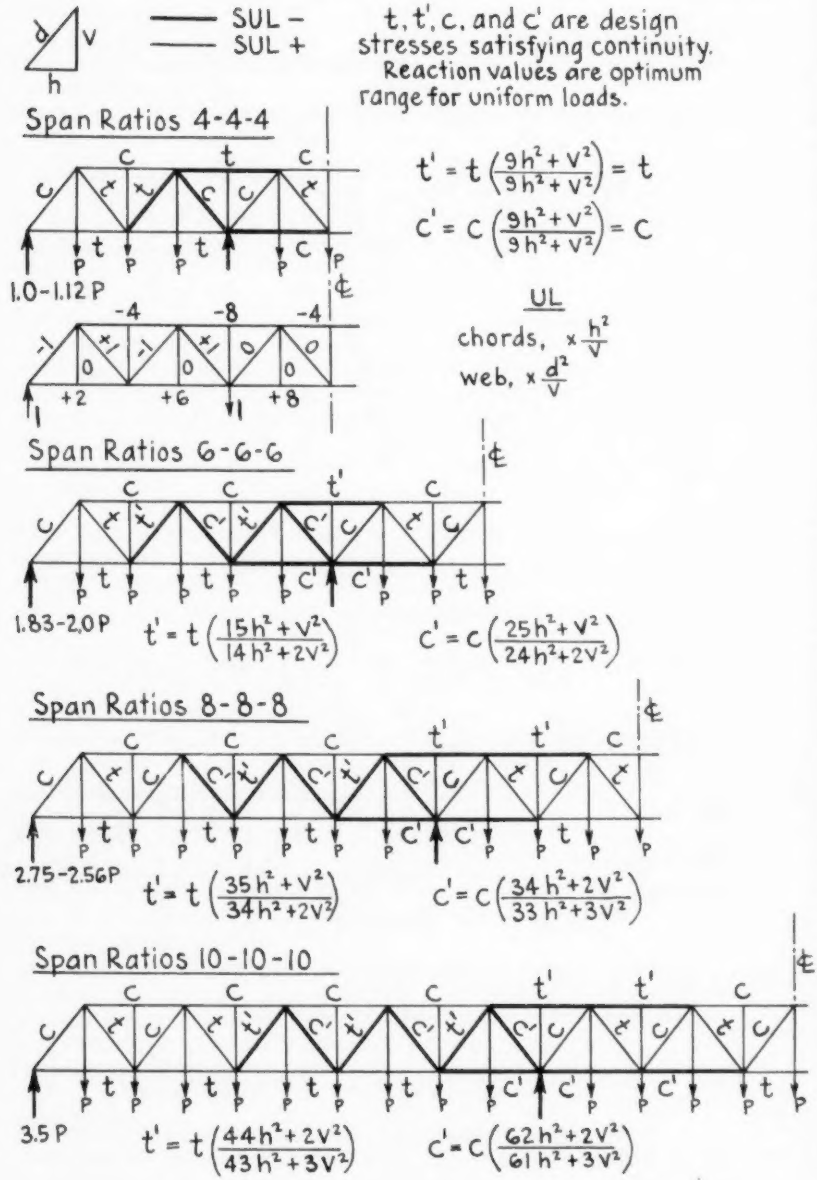


Fig. 5 Three Equal-Span Models.

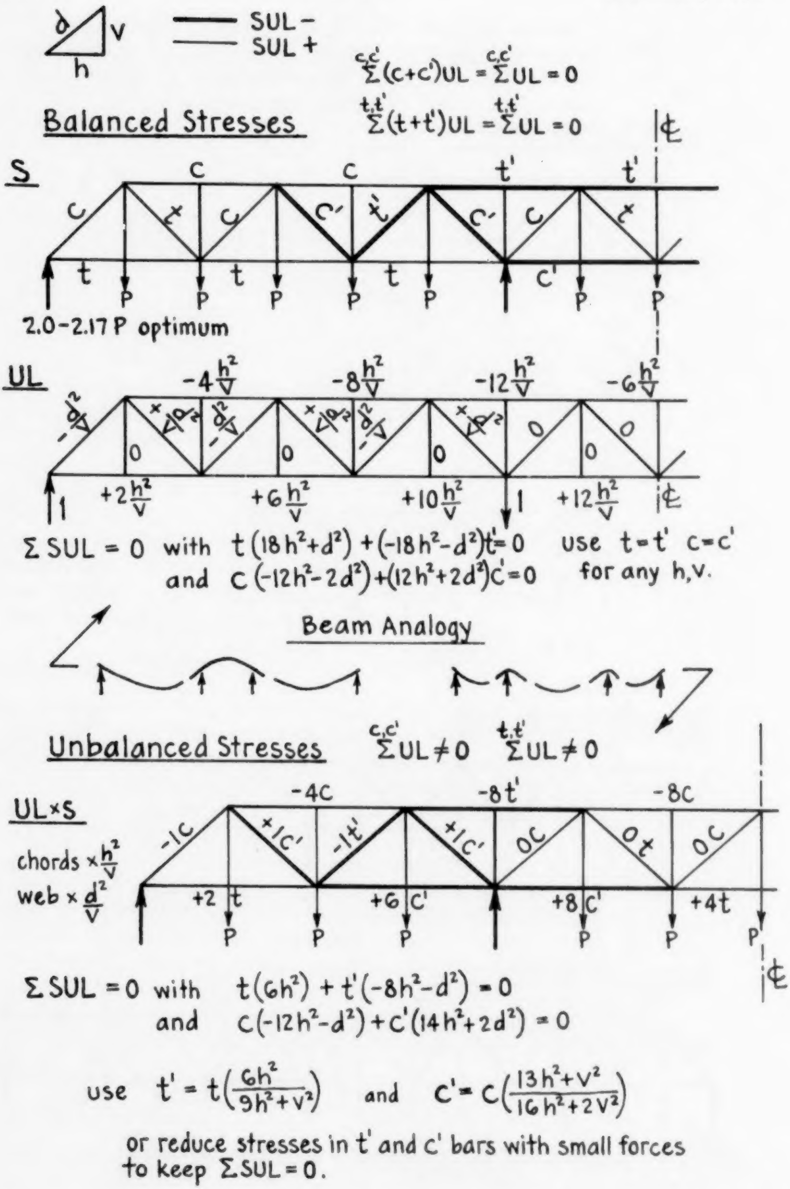


Fig. 6 Choosing Design Stresses for Continuity.

have been assumed and lengthy calculations made. It is expected that the influence lines might be modified for new trial structures during the trial-and-error process of reaching the final structure. In other words, the influence lines are dependent on the choice of the structure.

The direct-design method reverses the process; it fits the structure to the influence lines. Suitable influence lines for designing can be obtained from similar continuous beams, from similar trusses designed previously, or by scaling sketched influence lines judged suitable to the designer. If there are only a few live load combinations, influence lines are not even needed; the structure can be considered having a constant moment of inertia, and the reactions obtained by moment distribution.

Any selected influence lines can be used to obtain the final structure. The structure is designed from them and to agree with them. The influence lines are not tentative nor approximate for any one loading. They introduce only slight errors when used for many loading arrangements. If reasonable influence lines produce a very inefficient structure, changes should probably be made in the geometrical configuration of the structure—not in minor modifications of the influence lines.

Two-Span Truss

Fig. 7 shows three patterns of loading which typify: (1) full dead and full live load, (2) full dead plus live load on one span, and (3) full dead plus split live load. It is sometimes convenient to consider the dead and live loads simultaneously to simplify the design work by minimizing stress reversals in the bars.

Two comparative designs are carried out simultaneously in Fig. 7. Design (A) is from a sketched influence line assumed suitable for the center reaction, the values being shown at the top of the page. Design (B) is from reactions obtained by moment distribution. The three top sketches show only the maximum forces resulting from any of the three loading patterns.

The design stresses were obtained from the models analyzed in Fig. 4. Since for this particular structure $t = t'$ and $c = c'$, all tension bars were designed for 20 ksi, and all compression bars for 15 ksi. The final areas were then obtained by dividing the maximum forces by the design stress. Although the two designs require different areas throughout, the volumes of the trusses are substantially the same. Design (A) requires 580 volume units, and Design (B) 584 volume units.

The bottom sketch tabulates the actual maximum stresses in the bar areas of both designs, for all three of the loadings. The calculations were made by the usual analytical methods, of many times the length and complexity required for the designs. The discrepancies between the computed and the design stresses are negligible for both designs.

Three-Span Truss

Fig. 8 considers live loads only, in three critical patterns shown on the three top sketches. The reactions were obtained from influence line values for this truss with equal areas. The maximum bar forces for these three loadings appear on the appropriate one of the three sketches. The required areas were obtained by dividing these maximum forces by balanced compatible stresses obtained from the models of Fig. 5.

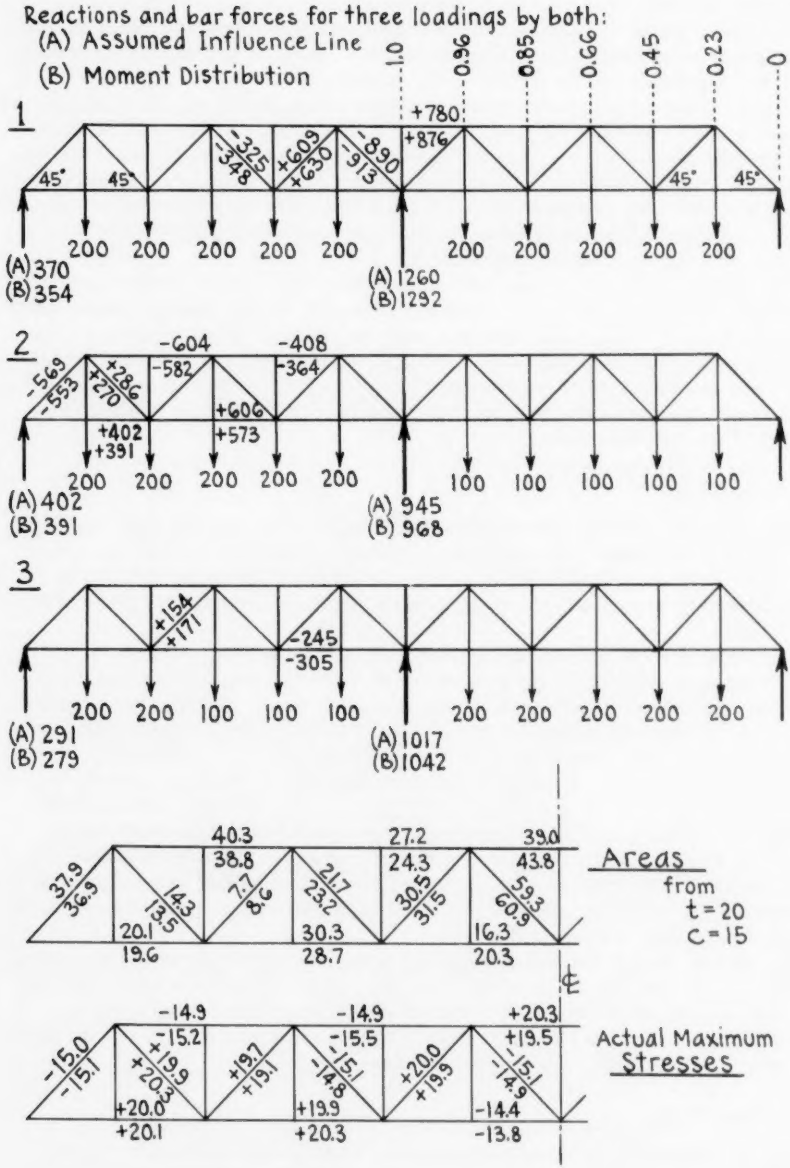


Fig. 7 Two Designs for Multiple Loadings.

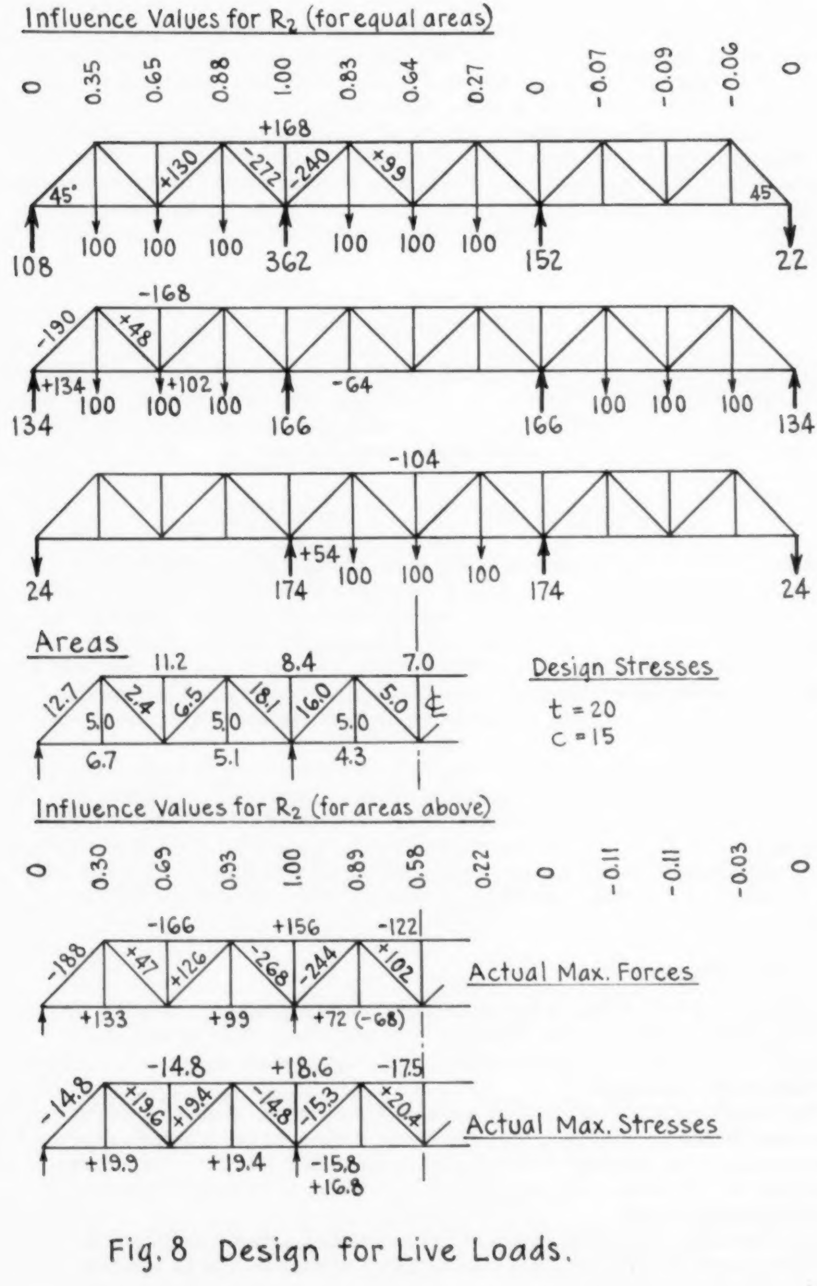


Fig. 8 Design for Live Loads.

Immediately under the sketch showing the design areas are the real influence line values for these areas, then the actual maximum bar forces, and finally the maximum stresses. All of these were calculated by the usual methods of analysis of a given structure. The calculated maximum stresses at the bottom of Fig. 8 prove that the areas from the direct design could be used for detailing the members. The small differences between the true stresses and the design stresses are caused by much more severe differences in the actual loadings here and in Fig. 5, from which the stresses were taken. Similar examples would confirm the fact that the designs are practically unchanged by rather extreme assumptions and by short-cuts which simplify the design work.

Many External Redundants

Highly redundant structures can be designed directly without simultaneous equations. Consider the most general structure, with n redundants, and with no deflection at any of the reactions. Applying virtual work, by imagining unit virtual loads at each of the redundants in turn, we have:

$$\Sigma S U_1 L = \Sigma S U_2 L = \text{------} \Sigma S U_n L = 0 \qquad (11)$$

In these equations, the stresses s are those with all redundants in place: the final, real stresses. The values of u are not at first evidently obtainable because of the indeterminacy of the remainder of the structure. However, the most general form of virtual work stated by Maugh(13) can be adapted to obtain a direct design. He states: "When displacements of any two points of a truss are known the movement of a third point can be obtained by applying virtual work to the simplest stable assemblage of members connecting the three points."

The basis of the application is that the deflection at any redundant is zero with respect to any two other reactions— redundant or not. The unit loads can be applied at each of the redundants, the reactions adjacent to each redundant then being the equilibrium reactions for the virtual load between them. The indeterminate truss has then been split into a series of overlapping simple trusses. The application can be explained further by the following example.

The General Method

The truss in Fig. 9 has three redundants, and no symmetry is given or expected in the reactions, bar forces or bar stresses. The method which follows is universal in its application to the most general structure, whatever its degree of redundancy.

The reactions were obtained by moment distribution. Sometimes these reactions should be adjusted, still satisfying statics, to get a judged superior distribution of bar forces, or to change the signs of a few of the bar forces. The first sketch shows only the signs of the forces, to keep the emphasis on the fundamental steps.

The four-span truss was split into three trusses, labeled A, B, and C. Each sub-truss is a simple span loaded with a virtual load at an interior point of a redundant. The trusses must overlap each other. Here truss B

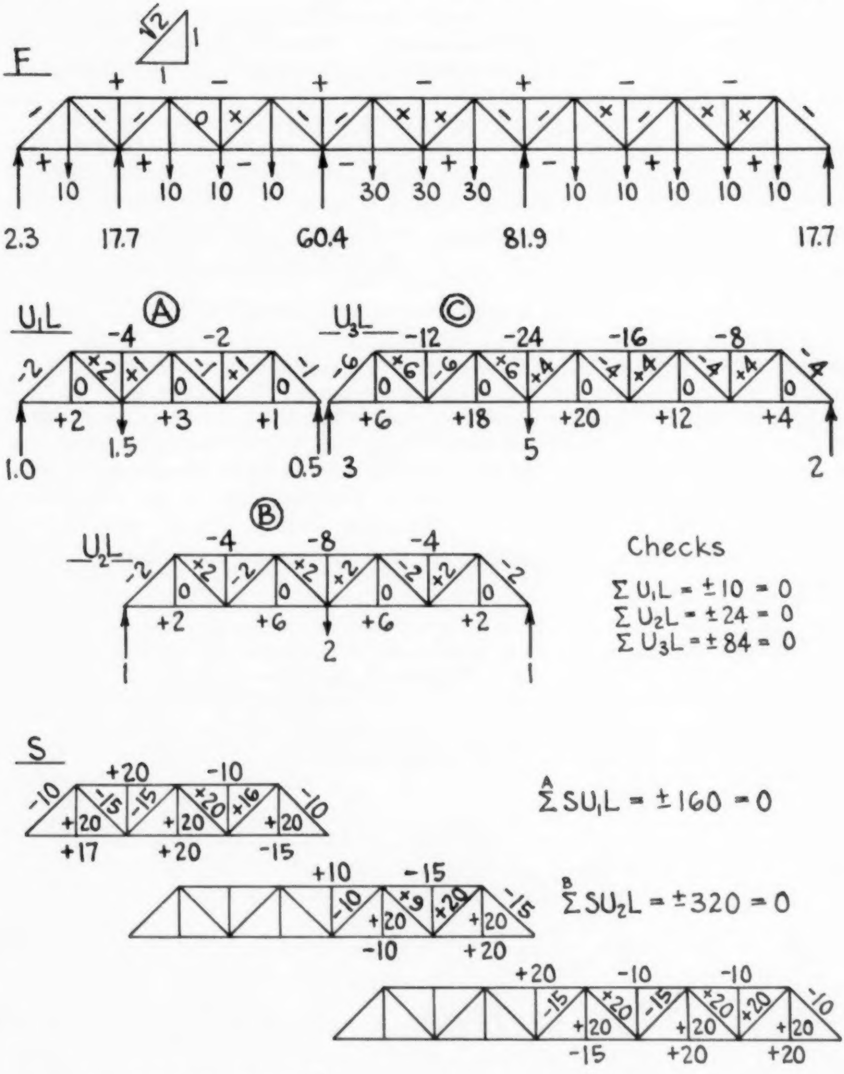


Fig. 9 Design of Sub-Trusses.

overlaps trusses A and C. The general rule for choosing the sub-trusses is to load each of the interior redundant reactions with a virtual load, and to support each load on the simplest stable determinate truss spanning between the adjacent reactions on each side of the load.

The values of uL were calculated for each sub-truss and labeled on the sketches. It seemed convenient to make the virtual loads 1.5, 2.0, and 5.0 to avoid fractions. This is permissible because each leads to a sum equal to zero. Working stresses were then obtained to satisfy the continuity of truss A, making $\sum^{A} su_1 L = 0$. The stresses for those bars common to truss B were transferred to truss B and multiplied by the new values of u_2L. The remaining stresses for truss B were then assigned to make $\sum^{B} su_2 L = 0$ over truss B. The process was repeated for the third sub-truss. In the complete structure, all of the sums of Equation (11) will be automatically equated to zero over the whole truss by this process. In this example, all of the stresses were assigned by trial, rather than by the method illustrated in Figs. 4 and 5. The absolute minimum volume of truss could be obtained by modifying the design process to include the principles of minimum weight.

Analytical Equations

The method using the sub-trusses can be expressed by the following equations:

$$\begin{cases} \sum^{1} SU_1'L = 0 = \sum^{1} \frac{FU_1'L}{AE} = \sum^{1} \frac{(F' + R_1U_1 + R_2U_2 + \cdots R_nU_n)U_1'L}{AE} = \\ \sum^{2} SU_2'L = 0 = \sum^{2} \frac{FU_2'L}{AE} = \sum^{2} \frac{(F' + R_1U_1 + R_2U_2 + \cdots R_nU_n)U_2'L}{AE} = \\ \sum^{n} SU_n'L = 0 = \sum^{n} \frac{FU_n'L}{AE} = \sum^{n} \frac{(F' + R_1U_1 + R_2U_2 + \cdots R_nU_n)U_n'L}{AE} = \end{cases}$$

$$\begin{cases} \sum^{1} \frac{F'U_1'L}{AE} + R_1 \sum^{1} \frac{U_1U_1'L}{AE} + R_2 \sum^{1} \frac{U_2U_1'L}{AE} + \cdots R_n \sum^{1} \frac{U_nU_1'L}{AE} = 0 & (12a) \\ \sum^{2} \frac{F'U_2'L}{AE} + R_1 \sum^{2} \frac{U_1U_2'L}{AE} + R_2 \sum^{2} \frac{U_2U_2'L}{AE} + \cdots R_n \sum^{2} \frac{U_nU_2'L}{AE} = 0 & (12b) \\ \sum^{n} \frac{F'U_n'L}{AE} + R_1 \sum^{n} \frac{U_1U_n'L}{AE} + R_2 \sum^{n} \frac{U_2U_n'L}{AE} + \cdots R_n \sum^{n} \frac{U_nU_n'L}{AE} = 0 & (12n) \end{cases}$$

These equations could be used for the analysis of a particular structure. The u-primes are obtained from one pound on the sub-trusses. The values of u are found by removing all redundants by the usual method. All sums here are taken over only the sub-trusses. All parts of Equation (12) transform into

Maxwell's equations if all of the u-primes on the sub-trusses are replaced by the usual u terms on the whole structure, and the limits of the sums are changed from the sub-trusses to the whole truss.

Settlement and Other Stresses

Stresses resulting from the settlement of a reaction can be obtained approximately with the sub-truss method. We will consider having a settlement at the n-th reaction of a highly externally redundant structure. The sub-truss would be a simple truss spanning between the next reactions on each side of the n-th redundant. The load would be a virtual one pound at the settled redundant. Virtual work then yields:

$$1 \times \delta = \sum_{n-1}^{n+1} u' \delta L = \sum_{n-1}^{n+1} \frac{u'uL}{AE} \tag{13}$$

where u is the virtual force due to one pound at the reaction, with all other reactions acting, u' is the statically determinate virtual force in the sub-truss; the summation being over only the sub-truss. The indeterminate values of u can be approximated by moment distribution, the only load being one pound at the point of settlement, with all other reactions supporting this load.

Equation (13) gives the settlement at the reaction for a one-pound force there. If the settlement is known or assumed, the reaction to produce it can be obtained immediately by proportion. The other reactions and the bar forces follow proportionately too, since they would be proportional to the values of u already obtained by moment distribution. The stresses would then be obtained from the forces and areas. The approximation should be a good one because moment distribution is quite satisfactory to obtain redundant reactions, these being little affected by the bar areas throughout the truss.

Other Applications

The calculation of stresses due to misfits and temperature, and the determination of influence lines are suggested by similar applications of the method.

Assigning Stresses by Trial

It is convenient to assign unbalanced stresses by trial in most direct designs. It is recommended that first the assumption be made that all assigned stresses will be the usual working values, carrying signs to agree with the assigned forces. By tabulating and summing the columns of positive and negative terms of suL, it will become clear that an arbitrary reduction must be made in some of the stresses which contribute items to the larger summation— to eventually make the positive and negative sums equal. These arbitrary reductions in some stresses should be made while viewing the forces, and anticipating the areas that will result everywhere over the truss.

Internal Redundants

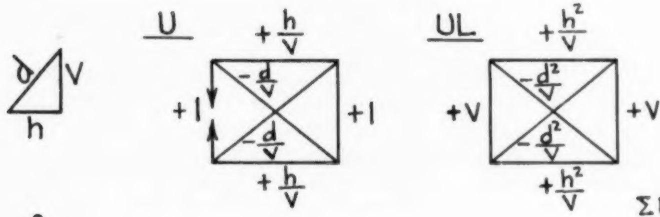
Internal redundants usually have the form of a pair of crossed diagonals. Fig. 10 is a fundamental analysis of such structural arrangements.

The values of u and uL are shown at the top of Fig. 10. For internal virtual loads, $\Sigma uL = 0$. Balanced stresses can be assigned only when the partial sums of uL equal zero over both the tension and compression bars. The equations show that balanced stresses can be obtained only when there is tension in one vertical, one diagonal and one horizontal member. The next series of sketches show cross-frames which could be assigned balanced stresses, and would be of minimum weight. From left to right, the first two would have different forces in the pairs of similar members. This would result in an awkward design. The third and fourth do not have this disadvantage. Their possibility of being practical structures is still small because of the inflexible position of the loads and the unloaded bars. The full structure at the bottom of the page cannot have all of its bars stressed to the allowable maximum values because all of the vertical members have the same type of stress: compression for these loads, and tension if the loads are on the bottom chord. The continuity of the structure can be satisfied only if some of the bars are understressed.

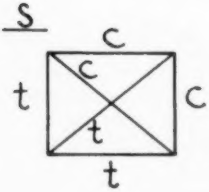
Approximate Methods

Fig. 11 is typical of the usual wind bracing structure. The desirable structure should possess such symmetry that loads (such as wind) could be applied from opposite directions, the structure being stressed identically with either loading. The diagonals should then carry equal forces. The equal division of the shear in a panel to the two diagonals results in the further advantage of equal forces in the chords in a panel, and equal forces in all verticals except those at the ends. The forces assigned to the structure in the first sketch of Fig. 11 would yield a desirable symmetrical structure if satisfactory stresses could also be assigned. It was previously shown on Fig. 10 that these panels would have to be unevenly stressed when three of the four members of the rectangular portion of a cross-frame have the same type of stress. An exact design can be made easily by assigning unbalanced stresses to one of the square panels to make $\Sigma suL = 0$. All other identical panels could have the same set of stresses assigned to them. The areas would then follow from the forces and these stresses.

A frequent approximation is to use the usual design stresses for all of the bars. Some justification will now be given for disregarding the incompatibility of the stresses. Of first importance is the fact that each indeterminate panel of Fig. 11 does not affect any of the other indeterminate panels in the calculation of the values of uL. Furthermore, each panel should have identical stress patterns, and each must satisfy the same deflection equation in the same identical way. This similarity of panels should not go unused. The second sketch of Fig. 11 shows the panels in a legitimate alternation of signs of uL. Since the gaps must close, it does not matter whether the unit load is tension or compression. The third sketch shows that (for an even number of panels) the values of uL cancel on the top chord, on the bottom chord, in the diagonals, and in the verticals. As far as the whole structure is concerned, its continuity is satisfied with design stresses having the signs of the forces expected. As far as each panel is concerned, the continuity is not necessarily satisfied.



$\Sigma UL = 0$



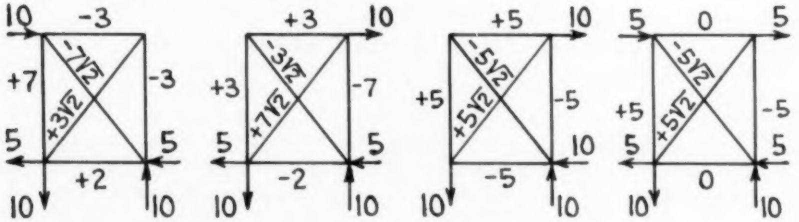
Balanced stresses only with t and c shared by the pairs of diagonals, horizontals, and verticals, since

$\Sigma SUL = 0$ with

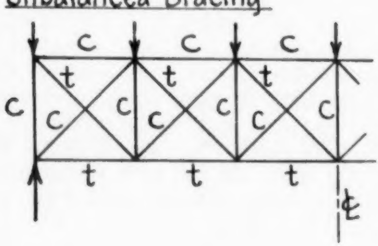
$$cUL = 0 = c\left[\frac{h^2}{v} + v - \frac{d^2}{v}\right] = c(d^2 - d^2) = c \times 0$$

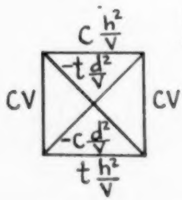
$$tUL = 0 = t[\quad " \quad] = t(\quad " \quad) = t \times 0$$

Balanced Cross-Frames



Unbalanced Bracing





SUL

$\Sigma SUL \neq 0$ since

$$tUL \neq 0; \quad tUL = t\left(\frac{h^2}{v} - \frac{d^2}{v}\right) \neq 0$$

$$cUL \neq 0; \quad cUL = c\left(\frac{h^2}{v} + 2v - \frac{d^2}{v}\right) \neq 0$$

Fig. 10 Analysis of Typical Bracing.

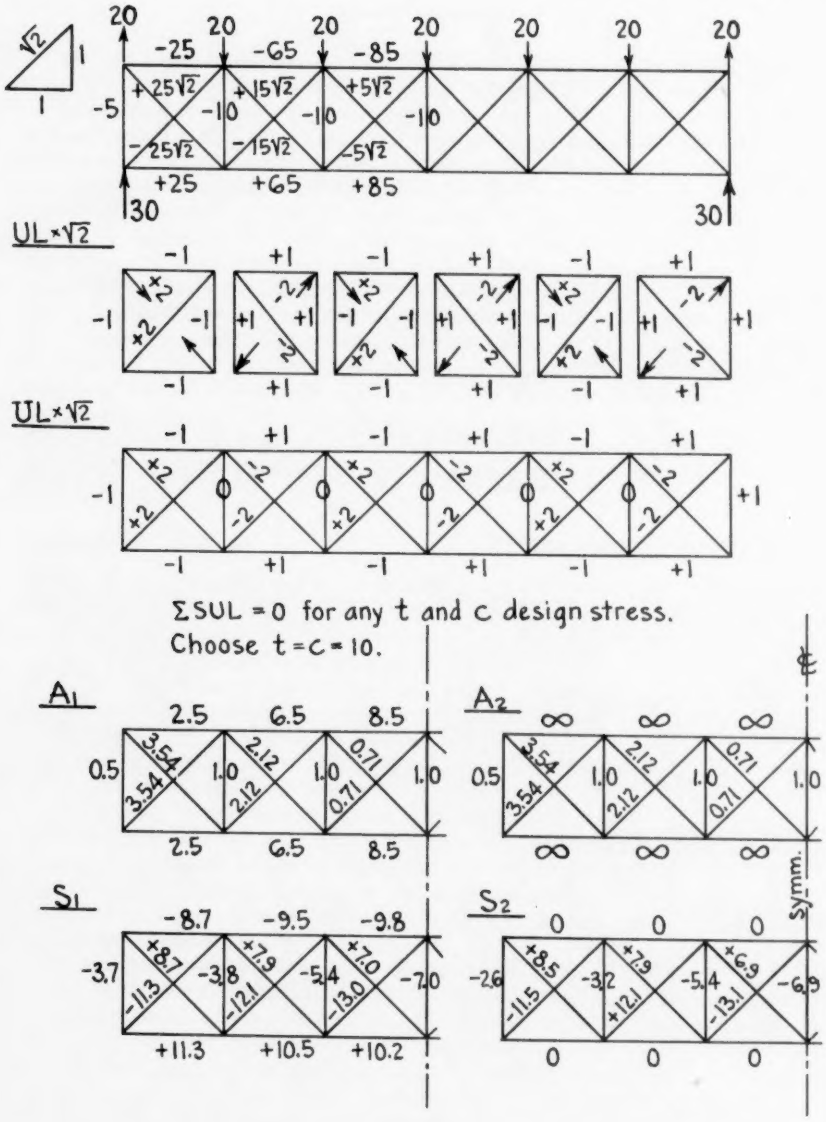


Fig. 11 Design of Wind Bracing.

The approximation is that each of a group of identical panels will satisfy its own continuity requirement if the entire structure, the aggregate of these identical panels, satisfies continuity. The physical application of this possibility is the loading of all unit loads simultaneously.

The truss of Fig. 11 was designed by this approximation. The last two line sketches actually are two designs: on the left, areas obtained by dividing all of the forces at the top of the page by a common design stress of 10 ksi; on the right, the same except that the chords were considered infinitely large as they might be when the bracing is connected directly to the flanges of a plate girder.

The last line shows the calculated exact stresses for the design areas above. These stresses were calculated by the usual method of simultaneous equations, using the areas that had been chosen for the designs. The approximation seems to have been quite satisfactory for the chords. All of the chord stresses are near the design stress, and understress on one chord is compensated for by an identical amount of overstress on the opposite chord. The average exact stress in the diagonals is also equal to the design stress, but the amounts of the overstress (and the equal understress) are only reasonably satisfactory. Nevertheless, the approximation of applying the unit loads simultaneously, of alternating signs, might be useful to a designer who uses judgment and experience in designing these highly redundant structures.

Internal and External Redundants

No complications arise if the structure is both internally and externally redundant. Fig. 12 shows the complete design of a structure having two external and six internal degrees of redundancy. To make the example as general as possible, no symmetry was anticipated in the final structure.

Statics was satisfied externally with reactions obtained by moment distribution. Statics was satisfied internally by dividing the shear in each panel equally to the diagonals. The final bar forces were thus obtained conveniently and shown on Fig. 12(a).

Fig. 12(b) shows how the structure was made determinate internally by removing the excess diagonal in each panel. One-pound loads were applied in place of the two external redundant reactions. The loads could be applied together because the values of u_1L and the stresses were both to be symmetrical about the center line. The values of u_1L were obtained, then the stresses on the right portion of Fig. 12(b) were assigned by trial to satisfy the continuity of this portion of the truss. In Fig. 12(c) the remainder of the stresses were assigned after the values of u_2L were obtained for each panel. Those stresses already assigned were multiplied by the newly introduced terms of u_2L, and the remaining stresses assigned, panel by panel, to close each of the panels. The stresses shown on the summary, Fig. 12(d), gives a completely compatible structure, with absolutely no discontinuities. The real stresses will also be exactly as assigned.

The final areas, Fig. 12(e) were obtained directly without solving simultaneous equations or making repeated analyses. It is also apparent at once that the final design is about the best that can be obtained. Minor changes to improve details and fabrication requirements could be made by revising some of the assigned stresses in Fig. 12(b) or (c), or by revising the assigned forces in Fig. 12(a). The reader who follows through this design in detail will

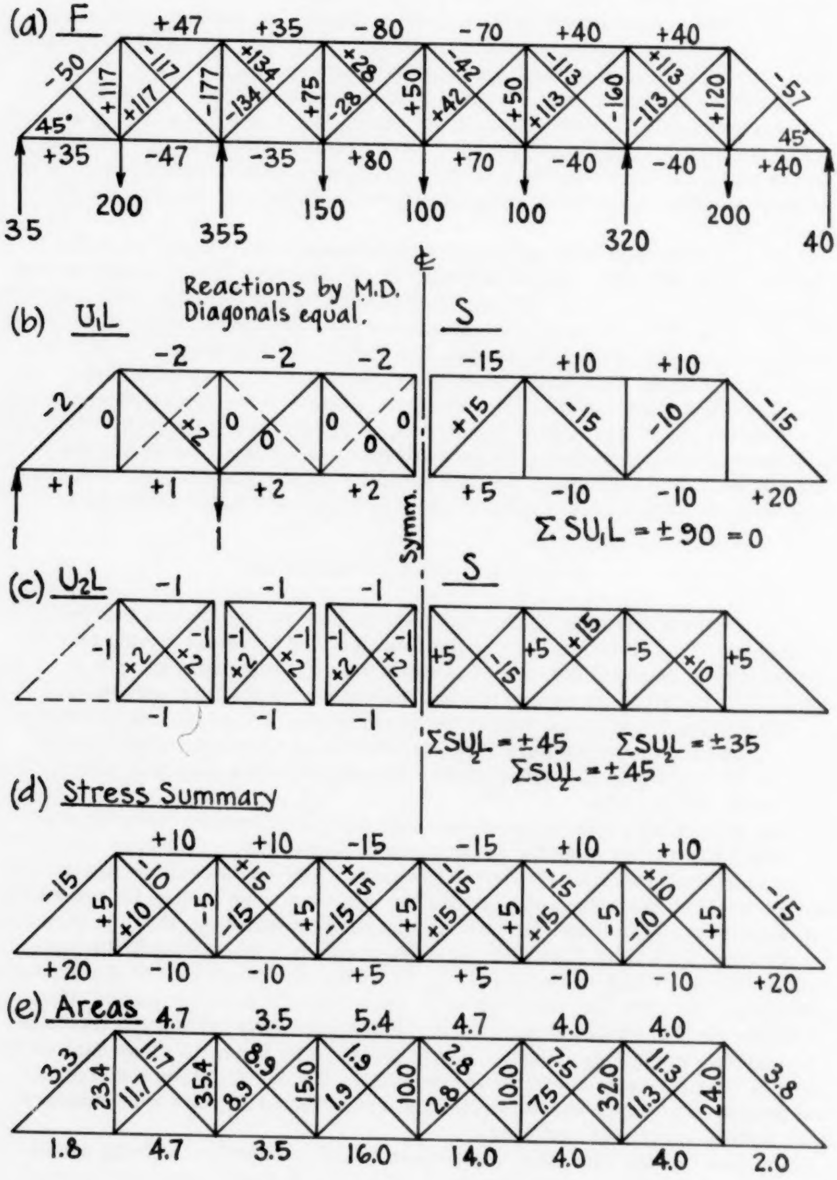


Fig. 12 Design with 8 Redundants.

notice that the sums of the u_1L are plus and minus one over the compression and tension bars, as close to zero as possible. But he will also see that the removal of the redundant $L_1 U_2$ would change the sign of the force in L_1L_2, and result in the ideal situation of both sums of u_1L being zero. The structure would be improved by omitting one of the redundants. The reader would also see that the centerline post prevents the assignment of maximum stresses to the entire structure. This example has been given to illustrate the simplicity of the idea of a direct-design, rather than to obtain the absolute minimum weight of truss, and to show how the method uncovers the geometric improvements that can be made to a clumsy structure.

Continuous Tied Arch

Practically the same methods were used to design the structure of Fig. 13. The redundant reaction and the tie force were assigned, and the bar forces were calculated by statics. Several trials were necessary to achieve a judged suitable distribution of forces. The values of uL for the structure were calculated, both with and without the tie. The stresses on the right portion of Fig. 13(c) were first obtained by trial to make $\Sigma su_tL = 0$. These stresses were then carried over to the whole structure shown on the left portion of Fig. 13(c), multiplied by u_rL, and the remaining stresses filled-in to make $\Sigma su_rL = 0$. The final areas are shown on Fig. 12(d).

Limit Design

It is well-known that an indeterminate structure has an excess of strength because of its redundancy. It is assumed that a structure will have reached its "ultimate load" when yielding or buckling has occured in a number of bars, one in excess of the number of red ndants. If trusses are to be designed on a limit-design theory, the direct-design method can be used. In fact, the elastic and limit theories can be merged.

The aim in assigning compatible stresses would be to assign the yield or buckling stress to as many bars as possible. Or if an elastic design had already been made, these stresses could be increased by multiplying them by the ratio of either the yield (or buckling stress) to the elastic design stress. If the structure had been assigned reasonably balanced stresses for the elastic design, collapse would occur in many bars simultaneously under the proportionately larger ultimate loads. Bar areas by both concepts would be immediately available: elastic areas, by dividing the assigned expected forces by the compatible stresses satisfying elastic specifications; limit areas, by dividing the expected forces increased by a load factor, by the limiting compatible elastic stresses. The state of the structure at the limit loading would still be elastic, with collapse impending by yielding and buckling simultaneously throughout almost the entire structure.

Advantages of Indeterminacy

A comparison of the relative advantages (and disadvantages) of indeterminate trusses over comparable determinate trusses is not appropriate here. The theoretical dead weights of alternative structures of each type is the only

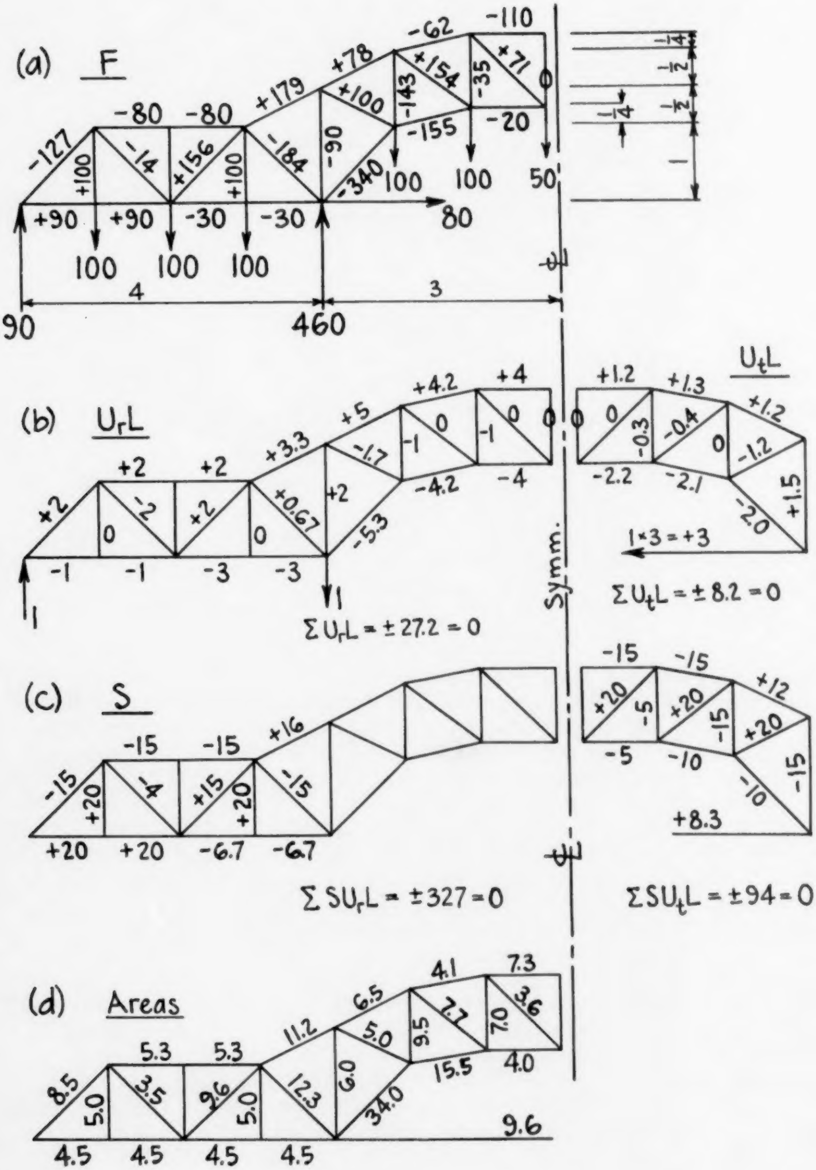


Fig. 13 Three-Span Tied Arch Design.

comparison pertinent to this paper. The comparison will be of simple trusses with parallel chords, to the same structures made internally indeterminate by adding redundant diagonals, or made externally indeterminate by adding members which connect the top chords. The extent to which the magnitudes of the compatible stresses can be balanced is an obvious requirement in the economy of an indeterminate structure. Whether the added redundants change any of the signs of the forces in the bars of the determinate portions is also of importance.

Internally Redundant

If the stresses can be balanced, and the redundants do not change the signs of any bar forces, the dead weight will not be changed by adding internal redundants. It has been proved that the same dead weight will result from all magnitudes of the redundants which yield forces with the same signs as a set of balanced, compatible stresses. The determinate truss can be considered having redundants of zero magnitude, where the zero amount is a boundary of the range of all optimum redundants which require the same amount of material.

If the stresses can be balanced, and the signs of the bar forces are changed by adding the redundants, savings in weight can be achieved by making the structure indeterminate. A careful study of the balanced cross-frames of Fig. 10 would disclose: (a) the same required weight when one diagonal is removed, without changing the signs of the forces in the remaining bars; and (b) more required weight in the determinate portion remaining after the removal of one diagonal, with changes in the signs of any of the remaining bar forces.

If the stresses cannot be balanced, and the signs of the bar forces are not changed by adding the redundants, the indeterminate structure probably will be heavier than the same structure without the internal redundants. The structure at the top of Fig. 11 can be used to prove this statement. Upon the removal of either redundant diagonal in each panel: (a) the remaining single diagonal must carry the identical shearing force as previously shared by a pair, (b) the two chords in each panel must have the same sum of forces with or without double diagonals, and (c) the verticals will have to carry less combined force when the structure is redundant. The only possible savings in weight are in the verticals of the indeterminate structure. But since the stresses must be unbalanced, some stresses being less than the usual working values, it is hardly likely that the added material required by low working stresses will be offset by the slight savings in the verticals.

If the stresses cannot be balanced, and the redundants change the signs of the forces in some of the bars, no general conclusions can be given concerning the relative economy of metal. A comparison would be needed for each particular structure. Since most internally redundant structures cannot have balanced stresses (at least for fixed loads), it seems likely that most internally redundant arrangements might require at least as much material as simpler, statically determinate arrangements.

Externally Redundant

The analysis of the relative weights is identical to that explained in detail for internally redundant structures. The ties between the top chords of the sub-trusses always change some signs of the bar forces in the otherwise

determinate trusses. The compatible stresses can usually be nearly balanced if one of the optimum reactions is assigned. Savings in weight are definitely possible. A quantitative study of the theoretical relative dead weights is desirable since externally redundant structures fit the two requirements for a savings in weight.

The model trusses of Fig. 5 are typical of structures for which approximately balanced stresses can be assigned, and for which a general comparison of relative weights can be made. Assuming the same design stresses then for both the determinate and indeterminate alternatives, the volume of metal will be proportional to $\Sigma|FL|$. The force in each diagonal is proportional to the shear; the force in each chord is proportional to the moment at the panel point opposite the chord; and the force in each vertical is unchanged by the indeterminacy.

Assuming the same dimensions in the two types of structures being compared, $\Sigma|FL|$ for the diagonals is proportional the area under both sides of the shear diagram. Precisely, FL in each member is $\frac{VL^2}{y}$, where y is the depth of the truss. Rearranging, $\frac{VL^2}{y} = (Vx)\frac{L^2}{xy} = \frac{V\,x}{0.5 \sin 2\alpha}$, where x is the horizontal panel point spacing, and α is the inclination of the diagonals. Of course, Vx is the area of the shear diagram between the panel points, and x would normally be constant over the truss.

For the chords, $\Sigma|FL|$ is proportional to the area under both sides of the moment diagram. Specifically, at each panel point, the forces in two chords will be $\frac{2M}{y}$. If y is constant for simplicity here, $\Sigma|FL|$ for both chords will be equal to $\frac{2A_m}{y}$, where A_m is the area under the moment curve. For the whole structure then:

$$\Sigma|FL| = \Sigma Py + \frac{2A_m}{y} + \frac{A_v}{0.5 \sin 2\alpha} = s\Sigma AL = s \text{ Vol.}$$

It should be noted that the areas under both curves must be considered positive, whether above or below the zero line. It is not the intention to discuss in detail the relative areas under the shear and moment curves for comparable determinate and indeterminate trusses. However, no elaborate calculations are necessary for the reader to prove that the area under the shear curve for the indeterminate structure will be equal or greater than the area under the shear curve for the determinate structures, and the area under the moment curve for the indeterminate structure will be less than the area under the moment curve for the determinate structures. Savings in weight in the indeterminate structures will be in the chords, these savings usually overbalancing any increased metal required in the diagonals. Finally, the relative economy of trusses is now possible by a study of the shear and moment curves for beams, for the terms just derived do not require the details of the members. They do require the usual working stresses; but where balanced stresses cannot be assigned, a suitable comparison with determinate curve areas is possible by increasing the shear and moment areas for the indeterminate structure by the ratio of the usual working stresses to the assigned average compatible stresses.

CONCLUSIONS

Methods have been presented to overcome the difficulties in the design of the most economical indeterminate trusses. Trial analyses of assumed structures have been replaced by methods which predetermine the optimum design. Simultaneous equations have been replaced by simple arithmetic. The emphasis has been on design, not analysis.

In summary, this is the direct-design method:

1) Redundant reactions are assigned by choice, usually by moment distribution, to satisfy statics.

2) Compatible stresses of the highest possible value are determined to satisfy continuity.

3) Final bar areas result by dividing the forces in equilibrium by the compatible stresses.

Some conclusions are:

1) The principle of potential work can be used to determine the most nearly balanced compatible stresses, to provide a check on statics, and to evaluate the weight and relative economy of indeterminate structures.

2) Compatible stresses can be obtained to satisfy a series of equations of the form: $\Sigma suL = 0$. Since $\Sigma uL = 0$, the separate sums over both the tension and compression bars can often be made zero, or nearly zero. Then the usual design stresses can be used.

3) With balanced stresses, all reactions within the optimum limits require the same minimum weight of truss. The reactions obtained by moment distribution are usually in this zone.

4) Influence lines can be selected, and the structure designed directly to fit them.

5) Multiply redundant structures can be designed as a group of statically determinate sub-trusses.

6) The final structure is often insensitive to reasonable approximations in the design method.

7) Externally redundant trusses can usually be designed with less weight than alternative determinate trusses. Internally redundant trusses will generally require an equal or greater weight than alternative determinate trusses, but savings are possible if the indeterminate truss is appropriate for the loads.

The direct-design method reveals the adequacy of a geometrical arrangement of bars. Improvements in the proportioning become evident. The best possible choice of members is determined readily. Experienced designers are provided with simple calculations to support their judgment. Inexperienced designers should find that the method discloses the basic action of indeterminate trusses — which is too often obscured by the enchantment of the classical simultaneous equations.

BIBLIOGRAPHY

1. "Analysis of Statically Indeterminate Structures," by Parcel and Moorman, John Wiley & Sons, Inc., New York, N. Y., 1955, p. 333.
2. "One Hundred Fifty Years Advance in Structural Analysis," by H. M. Westergaard, Transactions, Am. Soc. C. E., 1930.
3. "On the Application of the Principle of Virtual Velocities to the Determination of the Deflections and Stresses of Frames," by George Fillmore Swain, Journal of the Franklin Institute, Feb, Mar, Apr, 1883.
4. "Determination of the Stresses in Elastic Systems by the Method of Least Work," by William Cain, Transactions, Am. Soc. C. E., Vol. 24, April, 1891.
5. "Some Fundamental Propositions Relating to the Design of Frameworks," by Frank H. Cilley, Technology Quarterly, June, 1897.
6. "The Exact Design of Statically Indeterminate Frameworks. An Exposition of its Possibility, but Futility," by Frank H. Cilley, Transactions, Am. Soc. C. E. June, 1900.
7. "Design of Statically Indeterminate Trusses," by Albert Haertlein, Pub. Harvard Graduate School of Engineering, No. 183.
8. "A Strain Balance System for Analysis of Redundant Structures," by O. A. Baer, Proceedings, ASCE Separate 486, Structural Division, Sept., 1954.
9. "Influence Lines for Reactions of Continuous Trusses," by A. J. Pyka, Proceedings, Journal of the Structural Division, A.S.C.E., Vol. 82, March, 1956.
10. "Truss Analysis by Stiffness Considerations," by Harold C. Martin, Proceedings, A.S.C.E. Journal of the Eng. Mechanics Division, Vol. 82, October, 1956.
11. "Analysis of Multiple-Span Continuous Trusses," by B. C. F. Wei, Proceedings, Journal of the Structural Division, A.S.C.E., Vol. 83, March, 1957.
12. "Elastic Energy Theory," by J. A. Van den Broek, John Wiley & Sons, Inc., New York, N. Y., 1942, p. 34.
13. L. C. Maugh Discussion of "Truss Deflections by the Coordinate Method," by Kuang-Han Chu, Proceedings, A.S.C.E. Separate No. 54, January, 1951.

ADDITIONAL REFERENCE BOOKS

"Statically Indeterminate Stresses," by Parcel and Maney, John Wiley & Sons, Inc., New York, N. Y., 1926.

"Modern Framed Structures," by Johnson, Bryan, and Turneaure, John Wiley & Sons, Inc., New York, N. Y., 1932.

"Structural Theory," by Sutherland and Bowman, John Wiley & Sons, Inc., New York, N. Y., 1935.

"Theory of Continuous Structures and Arches," by Charles M. Spofford, McGraw-Hill, Inc., New York, N. Y., 1937.

"Theory of Statically Indeterminate Structures," by Fife and Wilbur, McGraw-Hill, Inc., New York, N. Y., 1937.

"Movable and Long-Span Steel Bridges," by Hool and Kinne, McGraw-Hill, Inc., New York, N. Y., 1943.

"Statically Indeterminate Structures," by L. C. Maugh, John Wiley & Sons, Inc., New York, N. Y., 1946.

"Elementary Structural Analysis," by Wilbur and Norris, McGraw-Hill, Inc., New York, N. Y., 1948.

"Theory of Modern Steel Structures, Vol. II – Statically Indeterminate Structures," by L. E. Grinter, Macmillan Co., New York, N. Y., 1949.

"Statically Indeterminate Structures," by C. K. Wang, McGraw-Hill, Inc., New York, N. Y., 1953.

"Fundamentals of Statically Indeterminate Structures," by Carl L. Shermer, Ronald Press Co., New York, N. Y., 1957.

"Indeterminate Structural Analysis," by J. Sterling Kinney, Addison-Wesley Publishing Co., Inc., Reading, Mass., 1957.

"Theory of Structural Analysis and Design," by James Michalos, Ronald Press Co., New York, N. Y., 1958.

Paper 1878 December, 1958 ST 8

Journal of the
STRUCTURAL DIVISION
Proceedings of the American Society of Civil Engineers

REVIEW OF LIMIT DESIGN FOR STRUCTURAL CONCRETE

C. W. Yu,[1] and Eivind Hognestad,[2] M. ASCE
(Proc. Paper 1878)

SYNOPSIS

The development of limit design of reinforced concrete structures is reviewed. Various theoretical approaches are discussed, and emphasis is placed on their relative merits. Codes of practice of countries recommending limit design are quoted. The importance of incorporating limit design into future United States practice is stressed, and approaches toward this aim are suggested.

INTRODUCTION

Inelastic behavior of structural concrete has played an important part in recent design recommendations both in this country and abroad. Non-linear relationships between stress and strain for concrete and steel influence the ultimate strength of individual cross-sections as well as the ultimate load capacity of an indeterminate structure as a whole. Codes of practice or specifications of many countries now base design recommendations on either one or both of these two related inelastic phenomena.

In Russia and Britain non-linear relationships are used in assessing the distribution of design moments in redundant structures as well as in determining the ultimate strength of sections. In Denmark and Norway moment redistribution by inelastic action is recognized in statically indeterminate structures, but design of sections is based on straight line theory and allowable working stresses. In Austria and Czechoslovakia the reverse is the case.

Note: Discussion open until May 1, 1959. To extend the closing date one month, a written request must be filed with the Executive Secretary, ASCE. Paper 1878 is part of the copyrighted Journal of the Structural Division, Proceedings of the American Society of Civil Engineers, Vol. 84, No. ST 8, December, 1958.

1. Development Engr., Structural Development Section, Research and Development Div., Portland Cement Assn., Chicago, Ill.
2. Mgr., Structural Development Section, Research and Development Div., Portland Cement Assn., Chicago, Ill.

It is clear, therefore, that engineers of the various countries have chosen different paths by which an eventual full consideration of plasticity in design is being approached.

Terminology

As a step toward unified terminology, and to clarify subsequent discussions in this paper, the following definitions are given as commonly used in the United States:

General Terms

Redistribution of Moments—results in a statically indeterminate structure from the formation of plastic hinges until the ultimate load is reached. As a result of the formation of plastic hinges, less-highly stressed portions of a structure may carry increased moments.

Yield Moment—in a member subject to bending, is the moment at which an outer fiber first attains yield point stress.

Plastification—is the gradual penetration of yield stress from an outer fiber toward the centroid of a section under increasing moment. Plastification is complete when the plastic moment is attained.

Plastic Moment or Ultimate Moment—is the maximum moment of resistance of a fully yielded cross section.

Plastic Hinge—is a yielded zone which forms in a structural member when the plastic moment is applied. The plastic hinge is capable of rotation as if the member were hinged, but the plastic moment is maintained at the hinge.

Hinge Angle or Hinge Rotation—is the angle through which the considered plastic hinge must rotate under the plastic moment before a sufficient number of further hinges form to develop a mechanism.

Rotation Capacity—is the angular rotation which a given member can sustain at the plastic moment without local failure at the plastic hinge.

Mechanism—is an articulated system of structural members connected by plastic and/or real hinges. The system is able to deform without finite increase in load until the deformations become so large that the equations of equilibrium are materially affected.

Ultimate Load or Plastic Load Limit—is the load attained when a sufficient number of plastic hinges have formed to transform either the whole or part of a structure into a mechanism. It is the largest load a structure can be counted upon to support.

Service Loads or Working Loads—are the loads the structure is expected to carry, permanently or temporarily, during its useful life.

Load Factors—are factors by which the service loads are multiplied to obtain the design ultimate loads. This choice of terms serves to emphasize the reliance upon load-carrying capacity of the structure rather than upon stress.

Structural Concrete Terms

Straight Line Theory—is a method of structural concrete design assuming a straight-line relationship between stress and strain of concrete and reinforcement, with some empirical modifications and adjustments to take into account certain inelastic characteristics observed in tests.

Ultimate Strength—is the largest moment, axial force, or shear a structural concrete cross section will support.

Ultimate Strength Design--is a method of design based on ultimate strength by inelastic action of structural concrete cross sections subject to bending, axial force, shear, bond, or combinations thereof. Ultimate strength design does not necessarily involve an inelastic theory of statically indeterminate structures. Evaluation of external moments and forces that act in indeterminate structures by virtue of dead and live loads may be carried out by the theory of elastic displacements, or by limit design and yield line theory.

Limit Design—is an inelastic theory of statically indeterminate concrete structures in which readjustments in the relative magnitude of internal moments and forces at various sections are recognized at high loads. Limit design does not necessarily involve a final design of cross sections on an inelastic basis. Sections may be designed by ultimate strength design, or by straight line theory.

Yield Line Theory—is a branch of Limit Design. It is a theory of reinforced concrete slab design based on inelastic action occurring in a pattern of yield lines, the location of which depends on loading and boundary conditions. Final design of cross sections may be carried out by ultimate strength design, or by straight line theory.

Structural Steel Terms

Plastic Design—is a structural steel design method which defines the limit of structural usefulness as the load attained when a sufficient number of plastic hinges have formed to transform the structure into a mechanism.

Plastic Modulus—is the bending resisting modulus of a fully yielded cross section. It is the combined statical moments about the neutral axis of the cross sectional areas above and below that axis.

Shape Factor—is the ratio of the plastic moment to the yield moment for a given cross section.

Working Stress Design—is a structural steel design method which defines the limit of structural usefulness as the load at which a calculated stress equal to the yield point is first attained at any point. Local stress raisers are usually disregarded.

Factor of Safety—as used in working stress design is a factor by which the yield stress of the material used is divided to obtain a working or allowable stress.

Present Status of Structural Concrete Design

During the past decade, experimental and analytical studies of inelastic behavior of structural concrete in the United States were purposely directed toward ultimate strength design under the guidance of the Joint ACI-ASCE Committee on Ultimate Strength Design which was organized in 1944. Relatively little emphasis was placed on limit design studies and yield-line theory.

This development resulted in the publication of a final report of the Joint Committee on Ultimate Strength Design in 1955.[30] Ultimate strength design was subsequently introduced as an alternative to the straight-line theory in the 1956 revision of the ACI Building Code (ACI 318-56)[38] The chosen approach toward recognition of inelastic behavior in design was clearly stated by the joint committee:[30]

"The Committee recognizes that in indeterminate structures, as ultimate load is approached, there is a readjustment in the relative magnitude of bending moments at various sections due to a non-linear relationship between load and moment with the moment in the more highly stressed sections increasing at a lower rate than in the sections less highly stressed. This inelastic behavior, commonly referred to as redistribution of moments in limit design, is important, but this report has been confined to design of sections and consideration of their ultimate strength.

"It is assumed that external moments and forces acting on a structure will be determined on the basis of the theory of elastic displacements. On the basis of this assumption stresses will remain within the elastic limits under service loads when proper load factors are used. For simple beams, the ultimate capacity equals the computed capacity. For indeterminate structures, the maximum moments at various sections are due to different load arrangements. Therefore, the maximum load capacity of a structure may be considerably greater than that indicated by the capacity at one section because of redistribution."

Having completed its assignment, the committee on ultimate strength design was discharged. A new Joint ACI-ASCE Committee on Limit Design was formed in 1957 to guide further developments of structural concrete design involving limit design. The review presented herein was made as a contribution to the activities of the Joint Committee on Limit Design.

Development of the yield-line theory is being guided principally by the Joint ACI-ASCE Committee on Design of Reinforced Concrete Slabs, and is not included in this review.

Plastic Design of Structural Steel

The development of design methods based on inelastic behavior of redundant steel structures preceded that of concrete structures. Therefore, it is helpful to review very briefly the work done on the former in order to appreciate discussions of the latter.

It is difficult to trace the origin of the concept of plastic design, but as early as 1914 G. von Kazinczy suggested the development of plastic hinges in continuous structures near ultimate load.(45) At the same time, it was noted that a fixed-end steel I beam could be designed for $wl^2/16$ for uniformly distributed load.(45) Following Professor Maier Leibnitz's observation of moment redistribution in fixed-end I beams in 1936, J. F. Baker experimented on small-scale fixed-end portals and developed the plastic hinge method of design. This was used in wartime construction of air raid shelters and other structures in Britain. After World War II, engineers throughout the world concentrated much effort in investigating the behavior of steel frameworks at ultimate load and in the development of practical plastic design methods. In 1948, Professor Van den Broek(3) wrote the book "Limit Design", and J. F. Baker published two articles on the semi-graphical design method by the plastic theory(5,7) in 1949. Then, Neal and Symonds introduced the more rigorous analytical approach, "The Inequality Method".(9,10) This was followed by the enunciation of "The Upper and Lower Bound Theorems" by Greenberg and Prager in 1951.(13) Neal and Symonds further suggested the "Rapid Calculation of the Plastic Collapse Load for a Framed Structure",(16)

while Horne extended the moment distribution method to the determination of plastic hinges.[20] Thus, a number of alternative methods of plastic design for steel frames have been developed and refined since 1945. These methods have been comprehensively grouped and treated in great detail in the form of lecture notes, design recommendations and other forms[8,12,19,33,39,44,45,48,50] for the benefit and use of practicing engineers. Sawyer[32] further developed an elasti-plastic design method for beams and frames, while Stevens published experimental and analytical data on the plastic design of arches.[49]

The approaches of the different methods as mentioned above differ, but they all recognize the following conditions as the requirements for collapse of an all-steel structure:

(1) Equilibrium Condition: Bending moment distribution must be in equilibrium with external loads.
(2) Collapse Mechanism Condition: A sufficient number of plastic hinges must exist to transform either the whole or part of the structure into a mechanism.
(3) The Yield Condition: Full plastic moment must nowhere be exceeded.

It is immaterial whether the problem is solved directly or indirectly. A design is considered valid when all the three numerated conditions are satisfied at the final collapse stage. It will be shown in the following that in reinforced concrete structures these conditions are also necessary, but they are not sufficient.

Limit Design of Structural Concrete

Plastic design of structural steel and limit design of structural concrete differ in two important respects.

Rotation Capacity.—In structural steel, the various design methods concentrate on the formation of plastic hinges sufficient in number to transform the whole or part of the structure into a mechanism, thus precipitating collapse. Little attention is paid to how much any one hinge section is strained before all the other hinges are formed. Such considerations are usually not necessary for structural steel[33] because under normal circumstances the ultimate strain of mild steel is greater than 15 per cent and far exceeds the strains acquired by moment distribution in any one section.

The ultimate strain for concrete in flexural compression is 0.3 to 0.5 per cent. Depending primarily on the amount of reinforcement, the ultimate strain in tension reinforcement varies from less than 0.5 to over 2 per cent. It is evident, therefore, that in limit design of structural concrete, rotation capacity of sections must be considered in greater detail than for structural steel. Furthermore, to avoid excessive flexural cracking, it is desirable to limit hinge rotations for structural concrete even when considerable rotation capacity is present after extensive cracking.

Distribution of Moment Resistance.—Unless coverplates or variable-depth sections are used, the positive and negative moment resistance of structural steel members are equal and constant along the entire length of a member. Thereby, plastic design of structural steel permits considerable cost savings as compared to elastic analysis. On the other hand, the inelastic structural

analysis must ensure that all three design conditions—equilibrium, collapse mechanism, and yield condition—are fulfilled, as mentioned earlier.

By varying the amount and location of reinforcement, the positive and negative moment resistance of structural concrete members can easily be made different, and the moment capacity can be varied along the length of a prismatic member. It is therefore conveniently possible to reinforce a concrete structure in such a manner that the distribution of moments at ultimate load capacity will be reasonably close to the moment distribution corresponding to elastic behavior. All plastic hinges necessary to form a mechanism will then form at practically the same load, and thereby the hinge rotations required are small. Similarly, it is possible arbitrarily to choose locations and plastic moments, for the number of hinges required to form a mechanism, in such a manner that the equilibrium condition is satisfied. The yield condition may then be satisfied by proportioning reinforcement to avoid yielding between the chosen plastic hinges.

Early Investigations

Early experimental work conducted to demonstrate moment redistribution in reinforced concrete beams dated as far back as 1920 when the results of the test on two fixed-end beams were reported by Deutsher Ausschuss für Eisenbeton.(1) It was concluded that the base line for the postive and negative bending moment diagram of fixed-end beams can be arbitrarily chosen.

In 1930 von Emperger also reported some tests on the moment redistribution of reinforced concrete beams, and he recommended that to design a beam with uncertain fixity it is only necessary to assume an arbitrary moment diagram in equilibrium with the external loads.

G. von Kazinczy, who originally conceived the possibility of the development of plastic hinges in structural steel, also conducted the first extensive series of tests demonstrating moment redistribution in reinforced concrete continuous beams. In 1933, he tested ten two-span continuous beams loaded at third-points.(1) The beams were intentionally "incorrectly" reinforced. Some were over-reinforced in span sections and some in support sections. To over-reinforce here means to provide a stronger section than is required by the theory of elasticity; the percentage of steel provided can still be small enough to permit ultimate strength of the section to be governed by yielding of the steel. He found that all beams failed when both the support and span sections reached their maximum moment capacity as evaluated by the ultimate strength theory of that period.

It should be emphasized that in all these early studies, plastic rotation and ultimate strength of all sections were governed by yielding of ductile reinforcing steel. No tests had so far been conducted in which moment redistribution had been a result of inelastic behavior of the concrete alone.

Work of Glanville and Thomas.—Glanville and Thomas(2) conducted a series of tests in 1935 to verify and demonstrate the redistribution of moments in reinforced concrete beams and frames as a result of yield in either the concrete or the steel.

The beams tested were two-span continuous beams loaded with a concentrated load in each span. In some of these beams, the support sections were reinforced to give steel yield, while in the others concrete yield. In the latter cases both the presence and absence of compression reinforcement were considered.

Except for the cases where compression reinforcement was present, it was found that moment redistribution continued until the span sections failed. In the exceptional cases, the support sections failed prior to full redistribution. In all these experiments, however, only one particular percentage of tension and compression reinforcement was investigated. Hence, no relationship could be established between the amount of steel used and the degree of redistribution attained.

For the experiments on frames, pin-ended single bay portals were chosen. Two cases were considered; (1) primary failure by yielding of steel, and (2) primary failure by crushing of concrete in the columns. It was found that for the former case full redistribution occurred, while in the latter case the columns failed first. Again, only one particular percentage of reinforcement was considered, and the results of these tests were by no means conclusive. Although they verified beyond doubt that moment redistribution occurred in reinforced concrete continuous structures, the results were only qualitative. Further analytical and experimental work was evidently necessary to enable designing engineers to predict with confidence the safe degree of redistribution in any one particular structure.

Work at Imperial College, London

The work done at the Imperial College under Professor A. L. L. Baker's direction is perhaps the most complete treatise on limit design, and the outcome is generally referred to as Baker's ultimate load theory. Its development is discussed in the following.

Baker's Early Findings.—In 1949, Baker put forward a trial and error method of computing the amount of moment redistribution in continuous beams.(4) He showed that even in the elasti-plastic stage the slope of a beam could be expressed as $\int M.ds/EI$, if EI values for the elastic and plastic stages are used appropriately.

Referring to Fig. 1, Baker showed that if the change of radius of curvature of a member was expressed in terms of the deformation of the concrete on the compression side of the neutral axis,

$$I = \alpha\, b d^3 \,(k_u^2 - k_2 k_u^3) \tag{1}$$

If the case of a two span continuous beam symmetrically loaded with a uniformly distributed load was considered, the moment diagram for the beam would be as shown in Fig. 2, where M_f and M_F were the free and redundant moments respectively. Applying the moment area principle,

$$\text{slope at } B = 0 = \frac{1}{\ell}\int \frac{M_F x}{EI}\,ds - \frac{1}{\ell}\int \frac{M_f x}{EI}\,ds \tag{2}$$

For a particular percentage of steel in the support section, k_u could be computed for the various sections along the length of the beam between the support and the first yielding section. The appropriate EI value could then be determined. The correct M_F value could be obtained by trial and error so that Eq. (2) was satisfied.

It was shown by this method that redistribution of moment due to primary crushing of concrete was not as effective as that due to primary yield of steel.

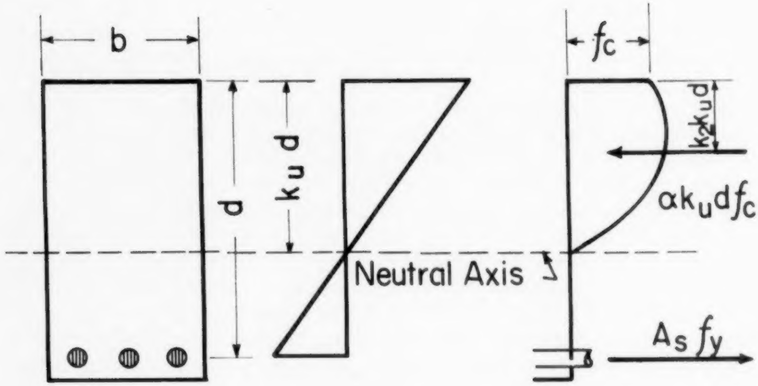

b
d
$k_u d$
f_c
$k_2 k_u d$
$\alpha k_u d f_c b$
Neutral Axis
$A_s f_y$
Section
Strain
Stress

Fig. 1

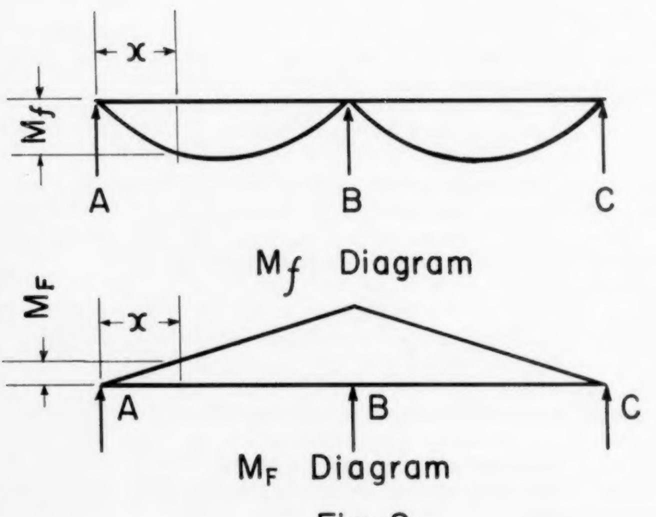

x
M_f
A
B
C
M_f Diagram
M_F
x
A
B
C
M_F Diagram

Fig. 2

Baker's General Equation.—Following the introduction of the "Plastic Hinge Theory" of structural steel by Professor J. F. Baker, Professor A. L. L. Baker postulated that similar treatment could be given to reinforced concrete structures, provided that strains at critical sections were checked.[14] In 1951 he proposed the following set of equations:

$$
\begin{aligned}
d_{01} + \bar{X}_1 d_{11} + \bar{X}_2 d_{12} + \cdots + \bar{X}_n d_{1n} &= -\theta_1 \\
d_{02} + X_1 d_{12} + X_2 d_{22} + \cdots + X_n d_{2n} &= -\theta_2 \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\
d_{0n} + \bar{X}_1 d_{1n} + \bar{X}_2 d_{n2} + \cdots + \bar{X}_n d_{nn} &= -\theta_n
\end{aligned}
\qquad (3)
$$

This set of equations involves the following assumptions:

(1) A structure n times statically indeterminate would develop n plastic hinges under increasing load prior to failure, and the plastic yield is concentrated at a hinge so that members between hinges remain elastic.

(2) The framework would become statically determinate, and the moments at the plastic hinges could be considered as remaining constant. (The magnitudes of the moments at these hinges are $\bar{X}_1$, $\bar{X}_2$ . . . etc., where 1, 2 . . . n denote the individual hinges)

θ_1, θ_2 . . . . . θ_n are the plastic hinge rotations, in radians, which are functions of concrete strains, at hinges 1, 2, 3 . . . . n.

d_{12}, d_{13} . . . . d_{1n} are the influence coefficients for hinge rotations of the framework when unit moment is applied at the hinge sections 1, 2, etc.

d_{01}, etc. are rotations at hinges 1, 2, etc. due to external load.

In subsequent years A. L. L. Baker and his team developed much analytical and experimental data to verify the validity of the equations.[18,21,24,25,26,29-31,40,42,46] A design method suitable for practical application took form gradually. In 1953 Baker introduced the trial and adjustment method, and at the same time he established some safe limiting θ values.[18] The fundamental principle of the method is to assign arbitrary values to $\bar{X}_1$, $\bar{X}_2$ . . . . . $\bar{X}_n$ in Eq. (3) and evaluate the θ values. If the θ values so obtained are less than the safe limiting ones, then the chosen $\bar{X}_1$, $\bar{X}_2$ . . . . . $\bar{X}_n$ values can be used, otherwise they are adjusted until the θ values are reduced to their permissible magnitudes. From the nature of the equations it is only logical to adjust $\bar{X}_p$ when θ_p is excessive.

The arbitrary values of $\bar{X}_1$, $\bar{X}_2$, $\bar{X}_3$ . . . . . $\bar{X}_n$ should be chosen from the following considerations.

(1) An economical distribution of moments at the ultimate stage should be achieved. That is, rectangular beams should have equal span and support moments due to vertical loads, while columns should have identical moments at the lower and upper ends due to lateral loads.

(2) Convenience of reinforcement detailing should be achieved. At sections where reinforcement is congested, moments can be reduced.

Characteristics of Baker's Equations.—Before proceeding further to discuss the subsequent development of Baker's method, it may be appropriate to discuss in fuller details the characteristics of the fundamental equations he established.

To utilize the set of equations, n hinges must first be chosen. It is not necessary to obtain the hinge positions by the methods developed for structural steel. It will be sufficient to choose as hinge points appropriate sections having maximum moments in the elastic case. Support sections of beams and junction sections of columns usually have maximum moments in the elastic stage. Such an arbitrary choice of hinge locations is possible because structural concrete can always be reinforced to fail in any desired manner. This is accomplished by having the framework members between chosen hinges designed to resist without yield the bending moments applied to them up to the formation of the nth hinge. In so doing, equilibrium and yield conditions are satisfied. Theoretically the full load applied to the structure will only cause the formation of n hinges in a structure n times statically indeterminate, and will not produce collapse. However, if the structure is reinforced as stated earlier, then a small increment of load will cause yield in other sections and transform the structure into a mechanism of (n + 1) or more hinges. This is illustrated in the following.

For a fixed-end beam loaded with a uniformly distributed load of w lbs per foot run (Fig. 3a), let the two hinges be chosen at A and B as in Fig. 3b. (A fixed-end beam is two times statically indeterminate: therefore two hinges are assumed.) The bending moment diagram as shown in Fig. 3b is compatible with the external loads. If the beam along length AB is reinforced to resist without yield this moment distribution, then yield will occur at PP' when w is $w + \Delta w$ or when the span moment is $\frac{(w + \Delta w)\ell^2}{8} - X$. A hinge will form at PP', giving a collapse mechanism as shown in Fig. 4. The collapse load for the structure is $(w + \Delta w)$ lbs per foot run, which for practical purposes can be considered as equal to w lbs per foot run.

Baker's fundamental equation is general and applicable to all statically indeterminate structures, neglecting fatigue. The main difference between the design of an n-times statically indeterminate reinforced concrete structure by the elastic and by the plastic theory is that in the former a set of n simultaneous equations, each equation involving n unknowns, has to be solved; in the latter n equations involving only ONE unknown in each equation have to be evaluated.

θ Values.—The basic relationship between the θ values as obtained from Eq. (3) and the behavior of the plastic hinge sections of a statically indeterminate structure was established by Yu,(22,25,27,30,31,42) who showed that

$$\theta = \int^{\ell_p} \frac{m}{(EI)_p} dx - \int^{\ell_p} \frac{m}{(EI)_e} dx \tag{4}$$

where ℓ_p = length of spread of plasticity along the longitudinal axis of the member

m = moment at sections along yield length,

$(EI)_p$ = EI value after yield and

$(EI)_e$ = EI value before yield.

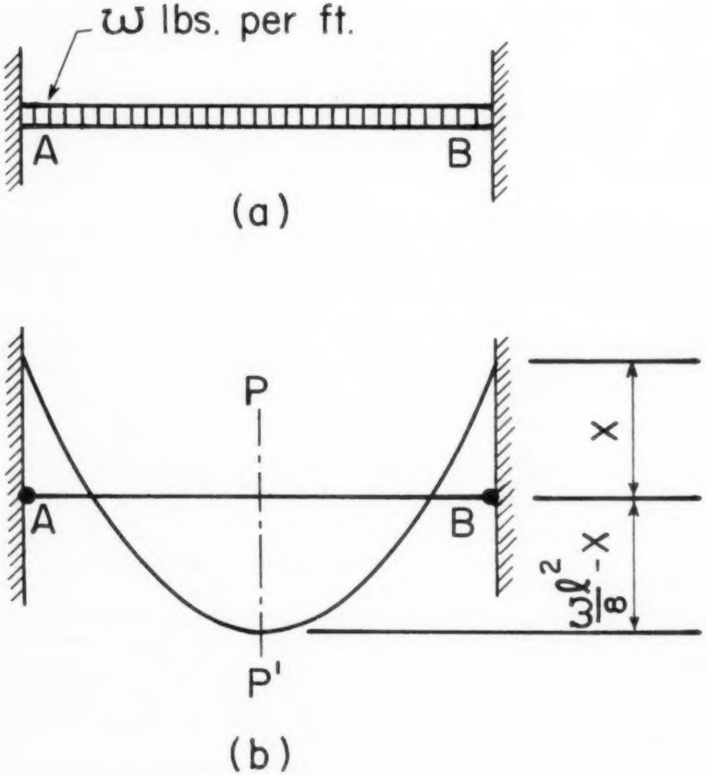


Fig. 3

Fig. 4

A. L. L. Baker derived and recommended the following expressions for θ

$$\theta = \frac{\varepsilon_u \ell_p}{k_u d} \tag{5a}$$

$$\theta = \frac{(\varepsilon_u - \varepsilon_s)\ell_p}{d} \tag{5b}$$

Expressions (5a) and (5b) are for tensile and compressive hinges respectively. (Tensile hinge means hinge section with tensile and compressive strain, while compressive hinge means hinge section with compressive strains only.) In the above expressions,

ℓ_p = length of yield

ϵ_u = plastic strain of concrete

ϵ_s = strain of reinforcement on the least stressed edge.

$k_u d$ = the depth of neutral axis at the instant concrete is crushed.

d = effective depth of section.

It should be noted that Eq. (4) can be reduced to (5a) or (5b) if the deformation at the elastic stage is neglected and the moment along the plastic length is assumed constant.

Baker recommended $\epsilon_u = .001$; $(\epsilon_u - \epsilon_s) = .001$; and $\ell_p = d$ as safe limiting values to be used in design. It should be noted that these recommended values are on the conservative side. Baker based his recommendations on results obtained from tests of statically determinate members. It has since been observed that ultimate strain of concrete in flexure of statically indeterminate members could reach much higher values.[34,35]

Rectangular Frameworks.—In applying Baker's fundamental equations to the design of a rectangular framework, which is one of the most common forms of reinforced concrete structures, Baker and Yu demonstrated that further simplification could be achieved.[18,24,25,29] Simple design formulas may also be evolved.

For rectangular frameworks, Baker[40,46] suggested that hinges may conveniently be assumed at the intersections of beams and columns. For this arrangement there are basically three different types of hinges:

(1) Hinges in the beams, classified as hinge type "b"
(2) Hinges at the top of the columns, classified as hinge type "a", and
(3) Hinges at the foot of the columns, classified as hinge type "c".

Adopting this system of hinges, general equations for θ can be written for a framework with any number of stories and any number of bays. By assuming moment values at the various hinges that will result in an economical distribution of moment at the ultimate stage, Baker has shown how graphs can be plotted to give directly θ values with respect to stiffness ratio between beams and columns.[46] For illustrations such graphs have been drawn for a 4-bay framework. The θ values so read off are checked against the permissible ones as explained in the general solution.

Necessity for an Approximate Elastic Solution.—All the steps so far discussed dealt with the assurance that a certain mode of moment distribution could occur without any of the hinge sections failing prematurely by excessive rotation. This obviously is one of the principle conditions to be satisfied in a reinforced concrete structure. However, there is yet another condition of no lesser importance to be considered. The structure has to be guarded against wide cracks, large deflections, and spalling of concrete at the service load stage. In order to check this, the approximate moment distribution at the working load stage must be known. This can be easily obtained from the plastic solution in the following manner.

Referring to Eq. (3), if the right hand side of the equation is 0, i.e. $\theta_1 = \theta_2 = \ldots\ldots = \theta_n = 0$, then $\bar{X}_1, \bar{X}_2 \ldots\ldots\ldots \bar{X}_n$ values that satisfy the equation would be the moments at the hinge sections if the frames were elastic. Hence, to obtain an elastic solution from a plastic one it is only necessary to adjust $\bar{X}_1, \bar{X}_2, \ldots\ldots \bar{X}_n$ values until $\theta_1, \theta_2, \ldots\ldots \theta_n = 0$. Since only approximate solutions are required, relaxation method can be conveniently employed as demonstrated by Baker, who strongly emphasized that in so doing it is not necessary to go into great refinement to obtain an exact solution because adequate strength is ensured from the plastic hinge calculation. It is only necessary to ensure that stresses likely to cause wide cracks or large deflections are not present in the working load stage.

As in normal relaxation operation, Baker recommends the following sequence of procedure:

(1) Reduce values of θ in their order of magnitude.
(2) Reduce values of θ at each hinge by adjusting the assumed bending moment at that hinge.
(3) Repeat the process until all the θ values are sufficiently small.

This is generally applicable, but when the number of bays or stories of a framework is great the process may become very lengthy. In view of this, Yu[21,25,46] developed a "Block Relaxation" procedure that converges more rapidly.

With the developments mentioned above, any framework can be designed by the "Limit Design" method to satisfy both the ultimate load and the working load conditions of the structure.

Chan's Study of Rotation Capacity.—The fundamental relationship between θ values as obtained from Eq. (3) and the actual development of plasticity in the hinge sections as established by Yu was verified experimentally and generalized analytically by Chan.[26,31] Referring to Eq. (4) it is evident that the length of yield is a function of:

(1) The moment-strain curve of the section, and
(2) The shape of the bending-moment diagram due to external load as shown in Fig. 5.

If the length of yield, ℓ_p, is determined from the above relationship, then θ is represented by the shaded area of the curve as shown in Fig. 6.

Chan conducted numerous experiments and concluded that if an appropriate quantity of "binders", such as spirals, is placed at the hinge section, the shaded area can be increased. In other words, the ultimate strain of the concrete could be controlled. It has been shown that concrete strain can be safely increased to as much as 0.01. With this high strain it is possible to accommodate all practical and economical modes of moment distribution in a redundant

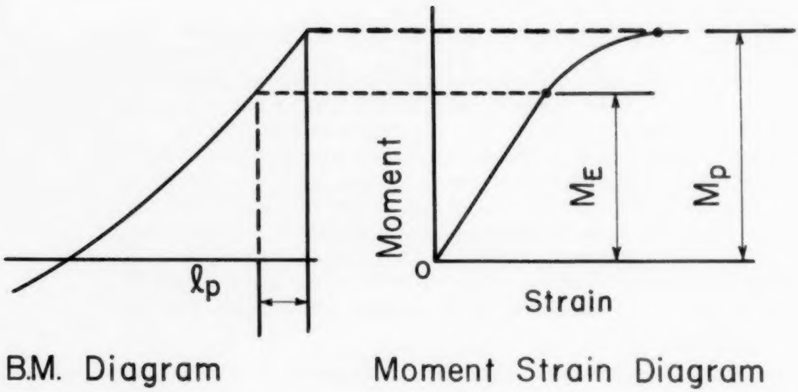

Moment
o
Strain
M_E
M_p
ℓ_p
B.M. Diagram
Moment Strain Diagram

Fig. 5

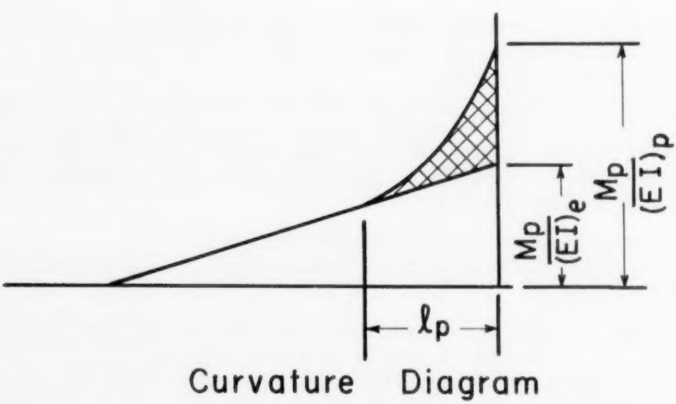

$\frac{M_p}{(EI)_e}$
$\frac{M_p}{(EI)_p}$
ℓ_p
Curvature Diagram

Fig. 6

structure. However, cracks and deflection under working load condition may often limit the permissible strains.

There is yet another point to bear in mind. The analysis presented by Chan is highly theoretical. His analysis is only valid if the state of stress near the support of a beam or junction of a column is the same as for the case of simple bending. This, however, is very doubtful, especially in a rigidly jointed structure where the sudden increase of rigidity near the junction causes a very complex distribution of localized stresses. Certain truss action could be developed, in which case it is far more realistic to assume constant strain and moment near the junction and use the simple expressions as given in Eqs. (5a) and (5b).

General Application of Baker's Theory.—Under certain circumstances it may be possible for the repetition of a load smaller in value than the ultimate load to be applied in a particular cycle a number of times to cause accumulative plastic strain in certain hinges and subsequently the failure of the structure.[9,10,43] However, from the statistic point of view it is argued that there is less chance for this "shake down" effect to occur than the full ultimate load occurring once. The limit design method as developed by Baker and his associates can thus be applied in general to all reinforced concrete structures.

The general solution for regular frameworks as described in preceding paragraphs does not limit itself to rectangular frameworks only. Yu[25] has developed a similar treatment for the case of pitched portals. He demonstrated that simple design formulas may also be derived, and that the approximate elastic solution may be obtained from the plastic case by a similar "Block Relaxation" procedure.

Other Recent Developments

Lee's Study of Rotation Capacity.—In 1955 L. H. N. Lee[37] suggested that, by assuming a stress-strain curve in concrete compression, a relationship could be established between moment and curvature which could be used as follows:

By differentiating the general equation of equilibrium,

$$f_c = pE_s\left(e_c \frac{de_s}{de_c} + 2e_s \frac{de_s}{de_c} + e_s\right) \tag{6}$$

where f_c = stress in concrete,

e_s = tensile strain in steel, and

e_c = compressive strain in concrete corresponding to f_c.

By measuring e_c and e_s from beam tests, a curve for f_c can be traced with respect to e_c. It was found that the stress-strain relationship could be approximated by $f_c = He_c - Be_c^2$, where H and B are constants and equal to $2f_{cm}/e_{cm}$ and $f_{cm}/(e_{cm})^2$, respectively, f_{cm} denotes the maximum compressive stress in the concrete, and e_{cm} its corresponding strain.

Since the strain of a fibre at distance η from the neutral axis is $\chi\eta$, where χ is the curvature, then the stress at the fibre under consideration is $H\chi\eta - B\chi^2\eta^2$. With this relationship the equation of horizontal force equilibrium becomes:

$$k^2\left(\frac{H}{2} - \frac{B\chi kd}{3}\right) = pE_s(1-k) \tag{7}$$

where k is the depth of the neutral axis. The moment of resistance expressed in terms of these parameters is:

$$M_r = bd^3\ k^2\chi\left[\frac{H}{6}(3-k) - \frac{B}{12}\chi kd(4-k)\right] \tag{8}$$

Either by eliminating k from Eqs. (7) and (8) or by actual substitution of measured k values from beam tests into these two equations, a relation between X and M can be derived. Once this relationship is known, distribution of moment due to plasticity for fixed-end beams, continuous beams and other simple structures can be determined in the conventional moment-area manner.

Ernst's Study.—Professor G. C. Ernst[27] suggested a more general approach following Lee's proposal. He restated the moment area theorems to include the behavior of structures in the inelastic range. Accordingly, a unit rotation diagram is used instead of the conventional M/EI diagram for the elastic case. The unit rotation diagram of a member with a definite yield length would be as shown in Fig. 7. In the diagram ϕ_e is the unit rotation at yield while ϕ_u is the value for the ultimate. Yield is assumed to spread along the entire length CD.

Using the following notations:

p = percentage steel

f_s = stress in steel

f_{av} = average concrete stress

k = ratio of depth of neutral axis to the effective depth of beam

e_c = strain of concrete at extreme fibre

e_t = strain of steel

then for equilibrium of forces

$$pf_s = kf_{av} \tag{9}$$

and for linear strain distribution

$$\frac{e_c}{e_t} = \frac{k}{1-k} \tag{10}$$

The unit rotation at any section = ϕ = $(e_c + e_t)/d$ where d is the effective depth. By eliminating k from Eqs. (9) and (10),

$$\phi d = e_c + e_t = \frac{e_c f_{av}}{pf_s} \tag{11}$$

Evidently ϕ_o and ϕ_u can be determined by substituting the appropriate values of e_c, ... etc. into Eq. (11). A stress strain curve, as first introduced in

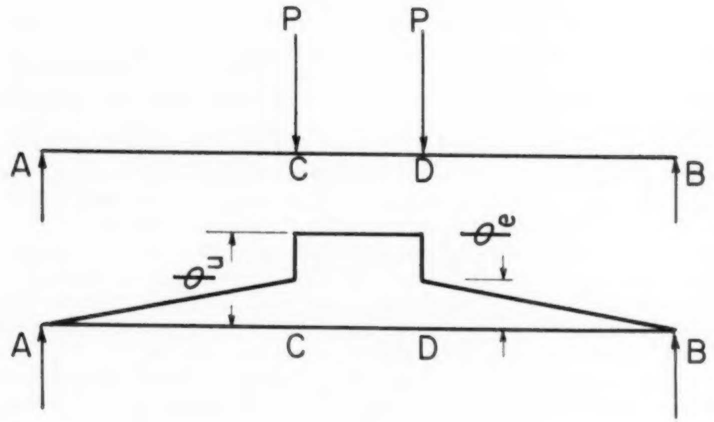


Fig. 7

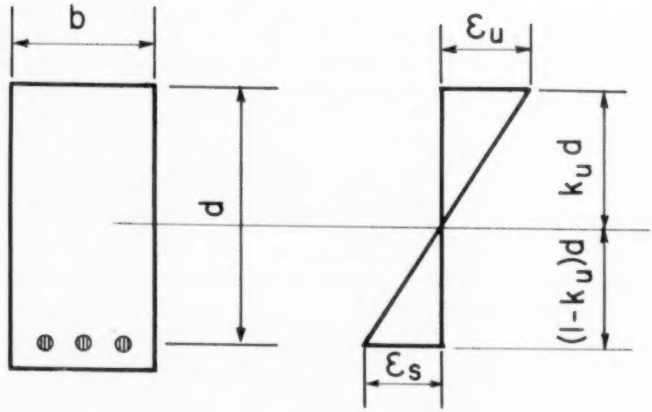


Fig. 8

Bulletin 399,[15] University of Illinois Experiment Station, was used by Ernst to obtain these values for both the yield and ultimate stages. For the length of yield, λ, Ernst assumed the total plastic rotation at a section to be $(\lambda\phi_u)$.

In design of indeterminate structures, plastic rotations are assumed at one or more of the sections in the structure to satisfy the various support conditions. If any of the $(\lambda\phi)$ values so obtained are negative, plastic rotation cannot take place at that section; and it should be assumed at some other section instead. By trial and error the magnitudes of all the plastic rotations could be determined.

Ernst[52] further conducted a series of experiments to investigate the amount of plastic rotation in simulated beam and column connections for both fast and slow loadings. The primary object of the tests was the study of plastic deformation available at failure. The experimental data would, of course, serve as a guide as to whether the plastic rotations obtained in his recommended analytical method are possible. The principal conclusion derived was that the amount of plastic rotation increases with decreasing steel percentage, confirming Baker's earlier analytical finding that redistribution due to primary crushing of concrete was not effective.

R. Gartner's Design Method.—Gartner[17] proposed a simplified solution to Eq. (3). He suggested that if working load and working stresses are used and if the values of $\theta_1, \theta_2 \ldots\ldots \theta_n$ are less than 30% of the corresponding values of $d_{01}, d_{02}, \ldots\ldots d_{0n}$, solution is satisfactory, provided that the hinges occur at the supports of beams and corners of columns. He based his proposal on the German recommendation that at support sections of beams the ordinary allowable stress of 705 psi may be increased to 1050 psi. (This is equivalent to 30% decrease of support moment.) This suggestion of Gartner is not exactly correct. Perhaps he considered that an experienced designer would very likely choose $\bar{X}_1, \bar{X}_2 \ldots\ldots\ldots \bar{X}_n$ values such that the ratio of these values to the actual elastic moments at the hinge sections is fairly constant, in which case the proposal is justified. On the other hand, should the analysis fall in the hands of an inexperienced designer who might simply assign $\bar{X}_1, \bar{X}_2 \ldots\ldots\ldots \bar{X}_n$ values at random, the solution could be on the unsafe side. This is a typical case, in which limit design does not involve the design of sections on an inelastic basis.

A few years later Gartner[51] recommended a more rigorous method of estimating θ values obtained by Baker's method. In the light of Chan's finding he also assumed that the length of plastic yield is a function of the external bending moment diagram.

If M_E = maximum elastic moment (due to either steel or concrete yield) of the yield section

M_p = maximum plastic moment (due to stress redistribution) of the yield section,

V = shear force at the yield section,

ϵ_s = strain in the steel at ultimate

ϵ_u = strain in the concrete at ultimate, and

ℓ_p = length of spread of plasticity, he established

$$\ell_p = \frac{M_p - M_E}{V}$$

He defined a steel hinge as one in which steel commenced to yield before the concrete, and a concrete hinge as one in which the concrete commenced to yield before the steel. For the former, $\theta = \epsilon_s \ell_p/(1 - k_u)d$ while for the latter $\theta = \epsilon_u \ell_p/k_u d$. For a section reinforced with a particular percentage of steel, M_E and M_p can be evaluated by using the appropriate k_u and α values (refer to Fig. 1), and hence the θ values can be checked. This evidently is a compromise between Baker's and Chan's method.

Marshall's Design Method.—W. T. Marshall(53) introduced a formula to evaluate θ involving the elastic and plastic moments of the section. By elastic moment of a section is meant the moment obtained from the elastic analysis. For a fixed-end beam (with supports at A and B) he suggested:

$$\phi_A = \frac{A_x}{EI\ell}\left(1 - \frac{M_P}{M_E}\right) \tag{12}$$

where A_x = moment of the area of the free bending moment diagram about support B.

E = modulus of elasticity of concrete, assumed to equal 1/n times the modulus of elasticity of steel.

M_p = the plastic moment assigned to the hinge section.

ℓ = the length of the fixed-end beam.

M_E = the elastic moment as explained above.

I = the moment of inertia of the transformed section, determined by using n = 40,000/ cube strength for long term work, and n = 20,000/ cube strength for laboratory work.

According to Marshall's formula, the elastic distribution of moment must be known first before θ values can be assessed. This method is not so practical for structures other than simple redundant structures.

Further Remarks on Limit Design Theories

The experimental and analytical work done in recent years, especially during the period after the war, has shown beyond doubt that moment redistribution does occur in reinforced concrete redundant structures. Although the research as related in the preceding paragraphs covers only the pioneer stage of the development, it must be said that one important step has already been put forward in the advancement of this branch of structural engineering. The proposed theories and methods have only been described briefly, but nevertheless the one put forward by Baker and his associates seems to be by far the more complete. It is a theory developed from sound assumptions, supported by rigorous mathematical analysis and numerous experimental verifications. In its light we see the explanations of some of the important engineering phenomena.

For example, in the work of Baker and associates, it was assumed that at the ultimate stage, moments at the top and bottom of the columns are identical due to the application of lateral loads to the framework. It has been shown that such a state of moment redistribution can be obtained. Structural engineers when considering laterally loaded frameworks usually solved the

problem by moment distribution method, or other approximate solutions. In view of the uncertainty of the EI values to be assumed in these solutions, many engineers consider it sufficiently accurate to assume the deflected shape of the structure to have points of contraflexure at the mid heights of columns. The forces and moments in the entire framework can thus be solved by simple statics. An analysis by elasticity will show that this assumption is incorrect, no matter what relative stiffness of members is assumed. However, the theory of limit design will prove that the intuitive judgment of these engineers is sound and correct.

Another problem of concern to structural engineers is that of the uneven settlement of the supports of a structure. Elastic theory will indicate that moments of considerable magnitude can be induced in the members for a normal small differential settlement. On the other hand, Baker and his associates have shown that this can be accommodated by the plastic rotation at certain hinge sections, without any increase of moment in any member. This accounts for the fact that in practical cases buildings can tolerate a certain amount of differential settlement, even though they are not designed for settlement.

It is often said on the basis of the elastic theory that in making some sections stronger than necessary the new structure will be at least as strong as the original one. It has been shown(25) that this is not necessarily true.

Baker and his associates have thus illustrated the importance of conceiving the behavior of a reinforced concrete structure at its ultimate. By so doing, the load carrying capacity of a structure is valued from the real characteristics and behavior of concrete.

Foreign Limit Design Codes

Certain foreign design codes recognize the inelastic behavior of concrete in their design recommendations. A brief discussion of these codes is given to illustrate the practical use of limit design.

British Code.—In the 1957 British code C. P. 114,(56) clause 213 states "Bending moments in beams and slabs should be calculated for the effective span and all loading thereon. The bending moments to be provided for at a cross section of a continuous beam or slab should be the maximum positive and negative moments at such cross-section, allowing in both cases, if so desired, for the reduced moments due to the widths of the supports, for the following arrangements of superimposed loadings:

(i) alternate spans loaded and all other spans unloaded;
(ii) any two adjacent spans loaded and all other spans unloaded.

Nevertheless, except where the approximate values for bending moments given in clause 313 are used, the negative moments at the supports for any assumed arrangement of loading may each be increased or decreased by not more than 15%, provided that these modified negative moments are used for the calculation of the corresponding moments in the spans."

Norwegian Code.—According to the 1957 Norwegian Standard 427B, it is recommended:(55) "In continuous slabs and beams in buildings which are designed by the theory of elasticity, that part of the support moment which is due to live load can be reduced by 25% without correspondingly increasing the span moment." (The maximum positive and negative moments are evaluated for similar conditions of loading as recommended by the British Code.)

Danish Code.—In the 1949 Danish Standard 411[6] the following method of designing continuous structures by the plastic theory is recommended: "In statically indeterminate structures the sectional forces may to some extent be arbitrarily chosen, provided that the maximum stresses at no point in such structures are less than 1/3 of the stresses computed by the elastic theory, or if it is possible to show by the plastic properties of the material that the sectional forces assumed to act simultaneously can actually do so, and with the assumed magnitudes."

Discussion.—The statements in both the British and the Norwegian codes are sufficiently clear; in the former a 15% redistribution of moment is permitted in both dead and live load, while in the latter a 25% reduction in support moment due to live load only is recommended. Although the figure of 25% in the Norwegian codes appears to be more generous than the British 15%, a rigorous comparison will show that the resultant redistribution of support moment permitted can be considerably greater in the British code, provided the ratio of live to dead load is small. In normal construction this is generally the case. For continuous beams it can be shown that the sum of the positive and negative moments according to the two codes is practically identical, and is independent of the live to dead load ratio. In other words, there is no difference in the load carrying capacity of any two beams carrying the same load, if one is designed according to the Norwegian practice and the other according to the British.

The 15% of redistribution is more than safe with respect to crack widths and deflection. Hruban and Svab[54] demonstrated that such a degree of redistribution could easily have taken place under working load due to the variation of rigidity along the lengths of members as a result of cracks in the concrete section, although the structure might have been designed primarily according to the elastic theory.

The Danish code, on the other hand, is rather obscure. Meyer and Moe[11] explained that the recommendation essentially implies that the moment chosen at a section should not be less than 1/3 of that obtained from the elastic theory. It should further be noted that no detailed procedure is given as "to what extent the moment of a section can be arbitrarily chosen." The recommended 66-2/3% of redistribution is indeed great. It should only be used, if at all, after careful analysis and consideration. It is intended by the code, according to Meyer and Moe,[11] that engineers not familiar with limit design should avoid its application, and engineers who have confidence in the theory should exercise careful judgment and thorough analysis in the choice of moment values.

Russian Recommendations.—The Russian Commission for Scientific Research[36,47] recommended the following slab and beam formulas:

$$\left.\begin{array}{ll} \dfrac{wl^2}{16} & \text{for span} \\[2ex] \dfrac{wl^2}{16} & \text{for support} \end{array}\right\} \text{ for intermediate spans of slabs and beams}$$

$$\left.\begin{array}{ll} \dfrac{wl^2}{11} & \text{for span} \\[2ex] \dfrac{wl^2}{14} & \text{for support} \end{array}\right\} \text{ for end spans of slabs}$$

$$\left.\begin{array}{ll} \dfrac{wl^2}{12} & \text{for span} \\ \\ \dfrac{wl^2}{12} & \text{for support} \end{array}\right\} \text{for end span of beams}$$

The above formulas are applicable for both uniform live and dead loads. In general, the moment arbitrarily chosen at any section, the recommendation further emphasized, should not be less than 70% of that obtained from the elastic theory. The reinforcement provided must be such that the ratio of the depth of the neutral axis to the effective depth of the section should not be less than 0.3. In other words, a section should not be too much under-reinforced. By observing the last two limitations the structure will be safely guarded against excessive cracks and objectionable deflections in the working load stage.

CONCLUDING REMARKS

It is evident that limit design of structural concrete is fundamentally a sound engineering design procedure. Limit design is a logical extension of ultimate strength design principles, and its introduction into practical design procedures must be expected to lead to better, more economical concrete structures:

(a) The maximum load capacity of statically determinate structures with sections proportioned by ultimate strength design equals the capacity computed by equilibrium considerations alone. For indeterminate structures, however, the maximum moments at various sections as calculated by the theory of elastic displacements are due to different load arrangements. Therefore, the maximum load capacity of the indeterminate structure as a whole may be considerably greater than that indicated by the ultimate strength of one section. By limit design, the moment redistribution involved is considered in design, and the maximum load capacity will equal the calculated capacity also for indeterminate structures.

(b) Limit design is simpler than design by the theory of elastic displacements. The former permits an <u>intelligent</u> arbitrary choice of redundant moments, while the latter requires solution of simultaneous equations by exact or approximate methods.

(c) A reduction of negative support moments by limit design would avoid reinforcement congestion. This would be an advantage particularly in buildings where negative beam reinforcement in two directions intersects the column reinforcement. By reducing beam moments, and thereby the amount of negative reinforcement, better concrete placement and compaction would become possible, and an improved concrete structure would result.

(d) Though high stresses may result from an elastic analysis, differential settlement and stress resulting from shrinkage and temperature change do not usually impair the load capacity of indeterminate structures. For very large differentials, however, rotation capacity of plastic hinges may become exhausted when live loads are applied, and the maximum load capacity may be reduced. Limit design permits a realistic

appraisal of the extent to which settlement and volume change affects load capacity.

(e) By proper choice of redundant moments, limit design permits control of flexural crack formation. For instance, if it is desirable for aesthetic reasons to avoid cracks in the columns of a rigid frame, the column moments may be reduced by reducing the negative beam reinforcement. Thus, flexural cracking may be shifted from the columns to the beam. Similarly, if exposure is a principal concern, flexural cracking may be shifted from exposed to protected parts of a structure.

Development of Practical Design Procedures

Advancement in the field of structural design and analysis must proceed with caution and deliberation. Structural design is fundamentally a combination of art and science molded by the designer's insight and experience as a builder. Whenever possible, new and improved design concepts should therefore be put to use gradually, so that departures from the type of structures for which extensive service records are available take place gradually.

Such gradual introduction of limit design may be achieved by initial consideration of continuous beams. A restricted departure from the moment distribution resulting from elastic analysis, similar to those now used in some foreign codes, must be seriously considered as a first step toward the use of limit design in American design practice. To develop specific design recommendations, experimental investigations of full-scale structural assemblies should be initiated. The test specimens and conditions used should realistically reflect conditions commonly encountered in practice. Emphasis should be placed on maximum-load capacity, rotation capacity, deflection, flexural cracking, and shearing strength.

Future Fundamental Research

Analytical Work.—The analytical approach to limit design of structural concrete should first be thoroughly studied. From the fundamentals of the theory, simplification should be made in the design method for simple redundant frameworks, so that design formulae will be readily available as in the case of continuous beams.

The evaluation of hinge rotation for three dimensional frameworks should next be developed. This is especially important in the case of frameworks with bow girders, which occur so often in modern construction.

The effect of buckling on the location of plastic hinges for a slender member requires more detailed analytical investigation.

More simplified solutions for the partial collapse and elasto-plastic cases are desirable.

Relationship should also be established between strain and hinge rotation of a member eccentrically loaded with respect to the two principal axes.

Experimental Work.—Tests on continuous beams should be conducted to observe the amount of moment redistribution as a function of reinforcement percentage. Crack distribution and deflection should be studied at all stages of redistribution. The spread of plasticity along the longitudinal axis of the member at the yielding zone should be observed. This will further clarify the complex nature of the state of stresses near the support of beams at yielding. A more precise evaluation of hinge rotation may become practical.

The effect of shear on hinge rotation should also be critically studied by tests. The presence and absence of web reinforcement should be the principal variable in this respect.

Frameworks should next be tested. The state of stress at a junction is three-dimensional in this case, and its effect on hinge rotation can be important.

REFERENCES

1933

1. Kazinczy, G. v., "Plasticity of Reinforced Concrete" ("Die Plastizität des Eisenbetons") Beton u Eisen. (Germany) V. 32 pp. 74-80. (in German)

1935-36

2. Glanville, W. H. and Thomas, F. G., "The Redistribution of Moments in Reinforced Concrete Beams and Frames", Paper No. 5061, Journal of the Institution of Civil Engineers (London), V. 3, pp. 291-329.

1948

3. Van den Brock, J. A., "Theory of Limit Design", John Wiley and Son, New York, 141 pp.

1949

4. Baker, A. L. L., "A Plastic Theory of Design for Ordinary Reinforced and Prestressed Concrete, Including Moment Redistribution in Concrete Members", Magazine of Concrete Research (London), V. 1, No. 2. June, pp. 57-66.
5. Baker, J. F., "The Design of Steel Frames", The Structural Engineer (London), V. 27, No. 10, Oct., pp. 397-431.
6. "The Danish Society for Engineers' Standard Specifications for Building Constructions No. 2, Plain and Reinforced Concrete", (approved as Danish Standard DS 411)" (Dansk Ingeniør-forenings Normer for Bygnings-konstruktioner, 2 Beton-og jernbetonkonstruktioner - Godkendt som Dansk Standard DS 411 af Dansk Standardiseringsråd) June, 60 pp. (in Danish)
7. Baker, J. F., "A Review of Recent Investigations into the Behavior of Steel Frames in the Plastic Range", Journal of the Institution of Civil Engineers (London), V. 31, No. 3. Jan., pp. 188-224.

1950

8. Horne, M. R., "Fundamental Propositions in the Plastic Theory of Structures", Journal of the Institution of Civil Engineers (London), V. 34, No. 6, April, pp. 174-177.
9. Neal, B. G. and Symonds, P. S., "The Calculation of Collapse Loads for Framed Structures", Journal of the Institution of Civil Engineers (London), V. 35, No. 1, Nov., pp. 2-40.

1950 (Continued)

10. Neal, B. G. and Symonds, P. S., "The Calculation of Failure Loads on Plane Frames under Arbitrary Loading Programmes," The Journal of the Institution of Civil Engineers (London), V. 35, No. 1, Nov., pp. 41-61.

11. Mayer, E. V. and Moe, A. J., "Comments to the Danish Society of Engineers' Specifications for Plain and Reinforced Concrete", (Kommentarer til Dansk Ingeniørforenings Normer for Beton - og jernbetonkonstruktioner af 1949) Teknisk Forlag (Copenhagen), 133 pp. (In Danish)

12. Neal, B. G. and Symonds, P. S., "Recent Progress in the Plastic Method of Structural Analysis," Journal of the Franklin Institute, V. 252, Nos. 5 and 6, pp. 383-407 and pp. 469-492.

1951

13. Greenberg, H. J. and Prager, W., "Limit Design of Beams and Frames", Proceedings of the American Society of Civil Engineers, V. 77, No. 59, Feb., pp. 1-12.

14. Baker, A. L. L., "Recent Research in Reinforced Concrete and Its Application to Design", Journal of the Institution of Civil Engineers (London), V. 35, No. 4, Feb., pp. 262-298.

15. Hognestad, E., "A Study of Combined Bending and Axial Load in Reinforced Concrete Members", Bulletin University of Illinois Engineering Experimental Station, No. 399, Nov., 128 pp.

1952

16. Neal, B. G. and Symonds, P. S., "The Rapid Calculation of the Plastic Collapse Load for a Framed Structure", Proceedings of the Institution of Civil Engineers (London), V. 1, Part 3, No. 1, April, pp. 58-71.

1953

17. Gartner, R., "Design of Indeterminate Structures by the 'Plastic' Method," Concrete and Constructional Engineering (London), V. 48, Nos. 1 and 2, Jan. and Feb., pp. 3-8 and pp. 85-94.

18. Baker, A. L. L., "Further Research in Reinforced Concrete, and Its Application to Ultimate Load Design", Proceedings of the Institution of Civil Engineers (London), V. 2, Part 3, No. 2, Aug., pp. 269-310.

19. English, J. M., "Design of Frames by Relaxation of Yield-Hinges", Proceedings of American Society of Civil Engineers, V. 79, Separate No. 322, Nov. pp. 322-1 to 322-12.

1954

20. Horne, M. R., "A Moment Distribution Method for the Analysis and Design of Structures by the Plastic Theory", Proceedings of the Institution of Civil Engineers (London), V. 3, Part 3, No. 1, April pp. 51-76.

1954 (Continued)

21. Yu, C. W., Correspondence on a paper, "Further Research in Reinforced Concrete and Its Application to Ultimate Load Design", Proceedings of the Institution of Civil Engineers (London) V. 3, Part 3, No. 1, April pp. 302-308.

22. Freudenthal, A. M., "Safety and the Probability of Structural Failure", Proceedings of the American Society of Civil Engineers, V. 80, Separate No. 468, Aug., pp. 468-1 to 468-46.

23. Craemer, H., "Plasticity and Limit Design in Reinforced Concrete", India Concrete Journal (Bombay), V. 28, No. 12, Dec., pp. 498-501.

24. Baker, A. L. L., "Recent Research at Imperial College", The Reinforced Concrete Review (London), V. 3, pp. 313-353.

25. Yu, C. W., "An Analytical Investigation of the Ultimate Load Design of Reinforced Concrete Framed Structures", Ph.D. Thesis, University of London (London).

26. Chan, W. W. L., "An Investigation of the Characteristics of Plastic Hinges in Reinforced Concrete", Ph.D. Thesis, University of London (London).

1955

27. Ernst, G. C., "Ultimate Slopes and Deflections—A Brief for Limit Design", Proceedings of the American Society of Civil Engineers, V. 81, Separate No. 583, Jan., pp. 583-1 to 583-21.

28. Müller, L. S., "Bending Moment Factors Based on the Theory of Plastic Hinges", Concrete and Constructional Engineering (London), V. 50, No. 4, April, pp. 149-154.

29. Baker, A. L. L., "High Building Frames and Foundations", Proceedings of the Institution of Civil Engineers (London), V. 4, Part 3, No. 2, August, pp. 228-245.

30. ASCE-ACI Joint Committee on Ultimate Strength Design, "Report of ASCE-ACI Joint Committee on Ultimate Strength Design", Proceedings of the American Society of Civil Engineers, V. 81, Separate No. 809, October, pp. 809-1 to 809-68.

31. Chan, W. W. L., "The Ultimate Strength and Deformation of Plastic Hinges in Reinforced Concrete Frameworks", Magazine of Concrete Research (London), V. 7, No. 21, Nov., pp. 121-132.

32. Sawyer, H. A., "Elasti-plastic Design of Single-Span Beams and Frames", Proceedings of the American Society of Civil Engineers, V. 81, Separate No. 851, Dec., pp. 851-1 to 851-29.

33. Beedle, L. S., Thurlimann, B. and Ketter, R. L., "Plastic Design in Structural Steel", Lecture Notes, Lehigh University.

34. Sahlin, S., "Effects of Far-advanced Compression of Concrete in Reinforced Concrete Beams Submitted to Bending Moments", Betong (Stockholm) V. 40, No. 2, pp. 135-149 (In Swedish).

1955 (Continued)

35. Nylander, H. and Salhin, S., "Investigation of Continuous Beams at Far-advanced Compressive Strains in Concrete", Betong (Stockholm) V. 40, No. 3, pp. 241-275 (In Swedish).

36. Tal, K. E., "Limit Design of Concrete and Reinforced Concrete Structures", State Publication (Moscow) 83 pp. (In Russian).

37. Lee, L. H. N., "Inelastic Behavior of Reinforced Concrete Members", Transactions of the American Society of Civil Engineers, V. 120, pp. 181-207.

1956

38. ACI Committee 318, "Building Code Requirements for Reinforced Concrete (ACI 318 - 56)", Proceedings of The American Concrete Institute, V. 52, May, pp. 913-986.

39. Jaeger, T., "The Fundamentals of Strength Analysis", (Grūndzuge der Tragberechnung) Der Bauingenieur (Berlin), V. 31, No. 8, Aug., pp. 273-291. (In German).

40. Baker, A. L. L., "Ultimate Load Design of Reinforced and Prestressed Concrete Frames", Symposium on the Strength of Concrete Structures, Cement and Concrete Association (London).

41. Hognestad, E., "Ultimate Strength of Reinforced Concrete in American Practice", Symposium on the Strength of Concrete Structures, Cement and Concrete Association (London).

42. Yu, C. W., Discussion on a paper, "Ultimate Load Design of Reinforced and Prestressed Concrete Frames", Symposium on the Strength of Concrete Structures, Cement and Concrete Association (London).

43. Rasmussen, B. H., "Incremental Collapse of Ordinary Reinforced Concrete Beams", Publication of the International Association for Bridge and Structural Engineering (Zurich), V. 16, pp. 439-456.

44. Dophin, J. W., Nelson, H. M. and Wright, D. T., "Plastic Design of Steel Structures", Lecture Notes, Royal Military College and Queen's University, Canada.

45. Baker, J. F., Horne, M. R. and Heyman, J., "The Steel Skeleton", V. 2, Cambridge University Press (England) 408 pp.

46. Baker, A. L. L., "The Ultimate Load Theory Applied to the Design of Reinforced and Prestressed Concrete Frames", Concrete Publication Ltd. (London), 87 pp.

47. Sachnowski, K. W., "Reinforced Concrete Structures" (Stahlbeton-Konstruktioner) Veb Verlag Technik (Berlin), 846 pp. (In German).

48. Jaeger, T., "Research and Design Methods in Plastic Design of Structural Steel", Part 1 and 2, (Tragfähigkeitsforschnung und Verfahren der Tragberechnung auf dem Gebiete der Stabwerke aus Baustahl, Teil 1 und 2) Bauplanung-Bautechnik (Berlin) V. 10, Nos. 7 - 8, pp. 266-280, 315-324. (In German).

1957

49. Stevens, L. K., "Carrying Capacity of Mild Steel Arches", Proceedings of the Institution of Civil Engineers (London), V. 6, Mar., pp. 493-514.

50. Merritt, F. S., "How to Design Steel by the Plastic Theory", Engineering News Record, V. 158, No. 14, Apr., pp. 38-43.

51. Gartner, R., "Design of Statically-Indeterminate Structures by the 'Plastic-Hinge' Method", Concrete and Constructional Engineering (London), V. 52, No. 5, May, pp. 161-174.

52. Ernst, G. C., "Plastic Hinging at the Intersection of Beams and Columns", Proceedings of the American Concrete Institute, V. 53, June, pp. 1119-1144.

53. Marshall, W. T., "Plastic Design in Reinforced Concrete", The Structural Engineer (London), V. 35, No. 7, July, pp. 243-251.

54. Hruban, K. and Svab, S., "Possibility of Saving Steel in Reinforced Concrete Frames", Transaction of Dr. E. Benes' Technical University, Brno (Czechoslovakia) 57 pp. (In Czech).

55. "Regulations for Design of Plain and Reinforced Concrete Works", Tentative Norwegian Standard No. 427 B. (In Norwegian).

56. "The Structural Use of Reinforced Concrete in Buildings", British Standard Code of Practice CP 114 (1957) June. 80 pp.

Journal of the
STRUCTURAL DIVISION
Proceedings of the American Society of Civil Engineers

ELASTI-PLASTIC ANALYSIS OF CONTINUOUS FRAMES AND BEAMS

Lawrence P. Johnson, Jr.,[1] A. M. ASCE and
Herbert A. Sawyer, Jr.,[2] M. ASCE
(Proc. Paper 1879)

SYNOPSIS

A practical analytical solution which considers both elastic and plastic flexural deformations is presented for continuous beams and frames. Method has limitations characteristic of limit design methods except that it determines strength as defined by the moment-curvature relationship as well as an ultimate moment, and it allows ready determination of deflections.

INTRODUCTION

As stated in previous papers,(1,2) for many loading conditions the strength of a flexural structure can be best evaluated by an analysis which considers the entire moment-curvature ($M-\phi$) relationship of its members. A practical solution of this type for single-span beams and frames has been previously presented.(2) For steel WF-section frames, Ang and Massard(3) have devised a method of determining the concentrated forces required to produce given inelastic deflections. This method, as was intended, is a valuable research tool for experimental work with impulsive lateral loadings. Baker and others at the University of London(4,5) have presented a method for reinforced concrete design which consists of satisfying sets of simultaneous equations, a different set usually being written for each loading condition.

The purpose of this paper is to present a method for the usual design problem for which loads rather than deflections are given and for which loads are often distributed. The solution is numerical rather than algebraic, using

Note: Discussion open until May 1, 1959. To extend the closing date one month, a written request must be filed with the Executive Secretary, ASCE. Paper 1879 is part of the copyrighted Journal of the Structural Division, Proceedings of the American Society of Civil Engineers, Vol. 84, No. ST 8, December, 1958.

1. Special Design Engr., Water Bureau, Metro. Dist. Comm., Hartford, Ct.
2. Prof. of Civ. Eng., Univ. of Connecticut, Storrs, Ct.

many of the techniques originated by Cross,[11] and the same elastic coefficients are used for all loadings. The method is applicable to continuous beams and frames made of any structural materials and with any number of plastic regions, provided that the moment-curvature relationships of all the structure's members are known.

For this method, a previously used[2] division of the flexural deformation or curvature into two parts was found convenient, one part being "elastic" and the other "plastic," Fig. 1. Then the total flexural deformation of a structure may be obtained by superimposing the (usually localized) plastic deformations and the elastic deformations. With total flexural deformations obtainable by this means, a solution of a structure is attained by satisfying the usual basic conditions: (a) the force system must satisfy statics; (b) deformations must be consistent with the force system; and (c) deformations must be consistent with the geometry of the structure and its restraints.

As is common for complex problems, a solution by successive approximations will be used, in each step of which a statically permissible set of moments is assumed, the corresponding plastic deformations are calculated, and, finally, the actual moments induced in the loaded structure from these plastic deformations are calculated and compared with the assumed moments.

In more detail, the solution will consist of the following steps:

1. Determine the elastic-solution moments.
2. Assume a statically permissible elasti-plastic-solution moment-diagram.
3. Compute the angle changes from the assumed moments caused by the plastic portion of the moment-curvature ($M - \phi$) curve, assuming the maximum diagram-moment to be the ultimate moment (M_u).
4. Apply these angle changes to the loaded structure, assumed to be otherwise elastic, and determine the resulting moments. (That is, add the elastic-solution moments of Step 1 to the elastic moments caused by the plastic angles.)
5. Compare these moments with the assumed moments. If practically equal, the structure is solved; if not, select intermediate values as the assumed moments for the next trial.

Moment-Curvature Relationships

The important problem of the determination, either theoretically or experimentally, of the $M - \phi$ relationship is beyond the scope of this paper. Reference (2) briefly discusses much of the past work which has been done on this problem. Recent papers by Mayerjak,[6] Hall and Newmark,[7] and Driscoll and Beedle[8] include additional information on $M - \phi$ relationships for steel members, and papers by Gaston, Siess, Newmark[9] and McCollister, Siess, Newmark[10] include such information for reinforced concrete members.

Ultimate Load vs. Ultimate Moment

It should be noted that an important assumption of this paper, as well as its predecessor,[2] is that the ultimate value of a slowly increasing proportional load system for a structure is attained when the ultimate moment (M_u) is first attained at any section of a structure.

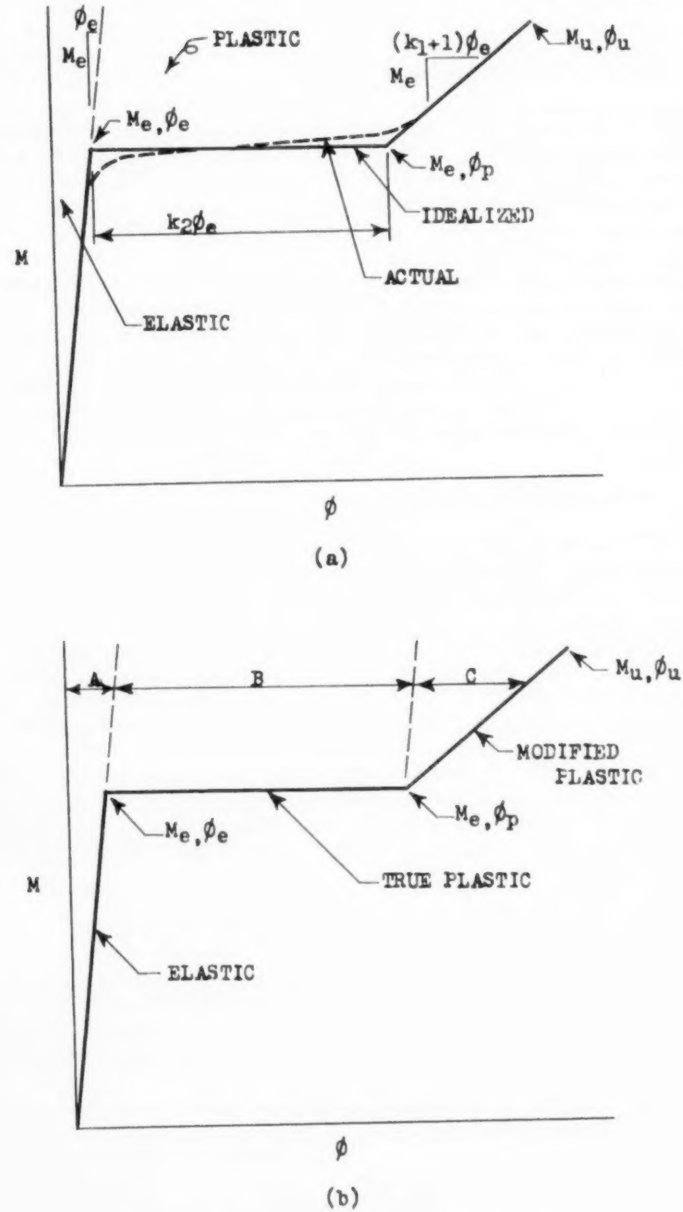


FIG. 1 - MOMENT - CURVATURE RELATIONSHIP

Actually, if the load on the structure were to exceed this value, two phenomena would occur which would have opposite effects on the resistance of the structure to the load. First, a load increase would increase the plastic angle for the initial-M_u region, thereby allowing the moments of the remainder of the structure to continue approaching their plastic mechanism-collapse values, thus tending to increase structural resistance. Second, the increase in curvature at the section of initial M_u will cause the moment at this section to decrease. This decrease, in itself, would tend to decrease structural resistance; also, it would cause a reduction of moment for the whole region (if the statically-determined moment-gradients or shears are not to be violated) which would tend to reduce the beneficial increase of bending angles of the region and thus oppose the just-mentioned approach of the remaining moments to their mechanism-collapse values.

Therefore, the change in structural resistance caused by these opposing phenomena could be positive or negative. Calculation of this change would be difficult and tedious, and, if calculated, the accuracy of the result would be so poor, because of many uncertain factors, as to divest it of significance. Also, any error from neglect of this change is always conservative. Therefore, the assumption of simultaneous attainment of ultimate moment and ultimate load is believed to be generally justified.

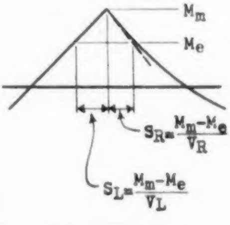
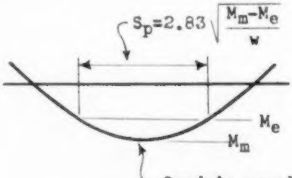
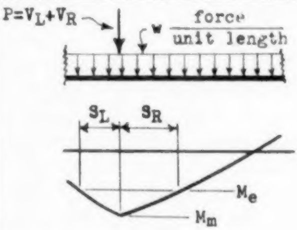
Evaluation of Angle Changes from Plastic Curvatures

An M - ϕ is shown in Fig. 1a which can be closely approximated by three linear relationships. Note that, knowing M_e and ϕ_e, the idealized curve is defined by three quantities, namely k_1, k_2, and the ratio M_u/M_e. (It should also be noted that, for the best approximation, M_e and ϕ_e usually exceed the true elastic-limit moment and curvature.) This idealized curve is useful for many steel WF and I sections, and, for relatively large values of k_2, the value of k_1 may be set at zero with little error. For most other types of members M - ϕ curves may be best and most simply approximated by appropriate values of k_1 and M_u/M_e, with k_2 equal to zero.

For any given moment, M, between M_e and M_u, as shown in Fig. 1b, the total curvature is made up of three parts: A, loosely termed the "elastic" portion; and B and C, "plastic" portions. The angle change in a certain plastic region of length S_p in a member caused by the plastic portions (B and C) will then be:

$$\theta = \int^{S_p} \left(\phi - \frac{M}{M_e}\phi_e\right) ds = k_2 \phi_e S_p + \int^{S_p} (M - M_e) k_1 \frac{\phi_e}{M_e} ds \qquad (1)$$

The integral of this expression involving k_1 has been evaluated previously(2) for certain common moment diagrams. These formulas (with k_2 equal to zero) are repeated for ease of reference in Fig. 2. The component of the plastic angle θ_p caused by the existence of k_2, which is $k_2\phi_e S_p$, is shown also in Fig. 2. If the term M_e/ϕ_e (or EI) is constant throughout a structure, it may be omitted from the computations for both the plastic angle (the resulting relative angle is denoted as θ'_p, as shown in Fig. 2) and the moments caused by it.

<table>
<tr><th>MOMENT DIAGRAM</th><th>VALUE OF PLASTIC ANGLE</th></tr>
<tr><td rowspan="3">
</td>
<td>$\theta_p = \frac{\phi_e}{M_e}\left\{k_2 M_e (M_m - M_e) + \frac{k_1}{2}(M_m - M_e)^2\right\}\left(\frac{1}{V_L} + \frac{1}{V_R}\right)$</td></tr>
<tr><td>If $k_2 = 0$;
$\theta_p \frac{M_e}{\phi_e} = \theta_p' = \frac{k_1}{2}(M_m - M_e)^2\left(\frac{1}{V_L} + \frac{1}{V_R}\right)$</td></tr>
<tr><td>If $k_1 = 0$;
$\theta_p \frac{M_e}{\phi_e} = \theta_p' = k_2 M_e (M_m - M_e)\left(\frac{1}{V_L} + \frac{1}{V_R}\right)$</td></tr>
<tr><td rowspan="3">
</td>
<td>$\theta_p = \frac{\phi_e}{M_e}\sqrt{\frac{M_m - M_e}{w}}\left\{2.83 k_2 M_e + 1.89 k_1 (M_m - M_e)\right\}$</td></tr>
<tr><td>If $k_2 = 0$;
$\theta_p \frac{M_e}{\phi_e} = \theta_p' = \frac{1.89 k_1}{\sqrt{w}}(M_m - M_e)^{\frac{3}{2}}$</td></tr>
<tr><td>If $k_1 = 0$;
$\theta_p \frac{M_e}{\phi_e} = \theta_p' = 2.83 k_2 M_e \sqrt{\frac{M_m - M_e}{w}}$</td></tr>
<tr><td rowspan="2">

V_L is shear to immediate left of concentrated load</td>
<td>$S_L = \sqrt{\left(\frac{V_L}{w}\right)^2 + \frac{2(M_m - M_e)}{w}} - \frac{V_L}{w}$
For S_R, replace subscripts L with R</td></tr>
<tr><td>$\theta_p = \theta_{pL} + \theta_{pR} = \frac{\phi_e}{M_e}(\theta_{pL}' + \theta_{pR}')$, where
$\theta_{pL}' = S_L\left\{k_2 M_e + \frac{k_1}{2}\left(M_m - M_e + \frac{w S_L^2}{6}\right)\right\}$
For θ_{pR}', replace subscripts L with R</td></tr>
</table>

FIG. 2

Moments Caused by a Plastic Angle

If a single-span fixed-ended beam of length L is subjected to any angle change,

$$\theta_p = \frac{\phi_e}{M_e}\theta_p' \qquad (2)$$

the location of whose centroid is known, the resulting change in the elastic fixed-end moments can be defined as:

$$\Delta FEM = C_m \frac{\theta_p'}{L} \qquad (3)$$

Considering the fixed-ended constant-EI beam shown in Fig. 3a, with an angle change θ_p either concentrated at C or with centroid at C, the moment M_o and the vertical force Y_o necessary at the elastic center to satisfy the restraint conditions are as shown.

The moment at B caused by M_o and Y_o is then:

$$M_B = M_o + \frac{L}{2} Y_o = (6K_b - 4)\frac{\theta_p'}{L}$$

and the moment coefficient for B is:

$$C_m = 6K_b - 4 \tag{4}$$

If end A were simply supported, the moment coefficient for B becomes:

$$C_m = 3K_b - 3 \tag{5}$$

Fig. 3b is a plot of these relationships. For members of variable EI, similar curves can be derived.

Usually the centroid of the plastic angle can, without serious error, be assumed at the point of the maximum moment. For extremely asymmetrical curvature diagrams, the actual location of the centroid can be computed.

For clarity of presentation, a geometric sign convention, as defined in Fig. 3, has been used through this paper. An algebraic sign convention similar to that used commonly in studies in mechanics of materials can be used if preferred. An algebraic sign convention of the slope-deflection type cannot be used unless some special provision is made for signs of intra-span moments.

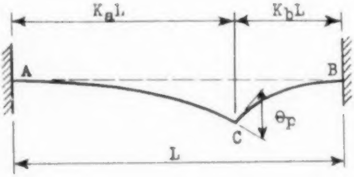
In members subjected to uniform loading, the location and magnitude of the maximum intra-span moment is not obvious. Fig. 4a shows such a member. Note, in Fig. 4b, that the maximum intra-span moment occurs at the point at which the slopes of the simple-beam parabolic diagram and the net moment diagram are equal, and its magnitude is the ordinate from datum to the parallel tangent. Fig. 4c is a plot of these solutions in terms of the ratio of end and maximum intra-span moments to WL/8, and K, the ratio of the distance to the maximum intra-span moment to the span length.

The moment effects caused by plastic bending in a continuous structure are found by distributing(11) the changes in the fixed-end moments of all members containing a plastic angle. Since several distributions are generally required, use is made of a procedure outlined by Looney(12) whereby a unit unbalanced moment is distributed throughout the structure, resulting in distribution coefficients which permit one-step distribution of any subsequently encountered unbalanced joint moment.

The following illustrative examples present the recommended procedure in detail.

Example 1 (Fig. 5), Three-Span Continuous Beam

As a simple illustration of the procedure, consider the structure shown in Fig. 5 with M - ϕ relationship constant and defined by k_1 negligible, k_2 equal to 12, and M_u/M_e equal to 1.05. Although this M - ϕ relationship approximates an experimental determination for a steel WF section,(8) it should not be regarded as representative of such relationships for all WF sections.



$$M_o = -\frac{\theta_p}{\int \frac{ds}{EI}} = \frac{-\theta_p EI}{L}$$

$$Y_o = -\frac{\theta_p(\frac{L}{2}-K_bL)}{\int \bar{x}^2 \frac{ds}{EI}} = -\frac{\theta_p L(\frac{1}{2}-K_b)}{\frac{L^3}{12EI}}$$

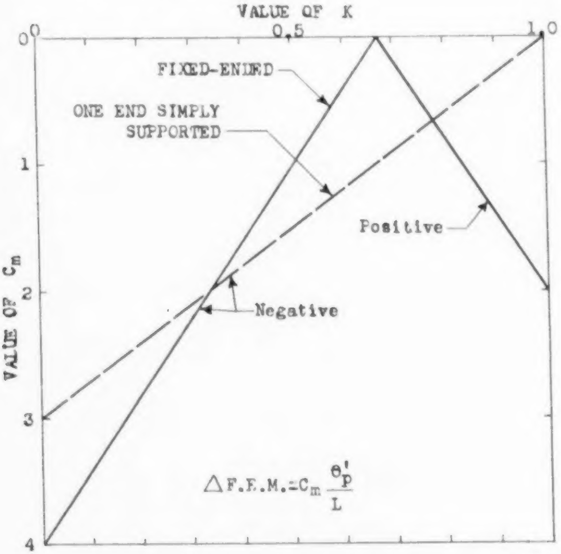
$$= \frac{-12\theta_p EI(\frac{1}{2}-K_b)}{L^2}$$

(a) - Beam Subjected to Concentrated Angle Change at C

Sign Convention:

Curvature	M	θ_p
	⌣	v
	⌢	^

A negative coefficient indicates a $\triangle$F.E.M. opposite to the sign of the applied curvature or θ_p.



(b) - Moment Coefficients

FIG. 3 - $\triangle$F.E.M. FROM θ_p' ON CONSTANT EI BEAMS

Distribution coefficients for unbalanced joint moments and the subsequent elastic solution of the structure for the given loads are shown immediately above the first elasti-plastic trial.

Consider "elasti-plastic trial # 3" as the typical cycle. Increments in moment, $\Delta M'_3$, assumed with the aid of information from the previous trial, are added to the assumed moments of the previous trial to give the assumed moments, M'_3, for this cycle. The values of the end shears, V, found by statics from the assumed moments, are then listed for convenience in

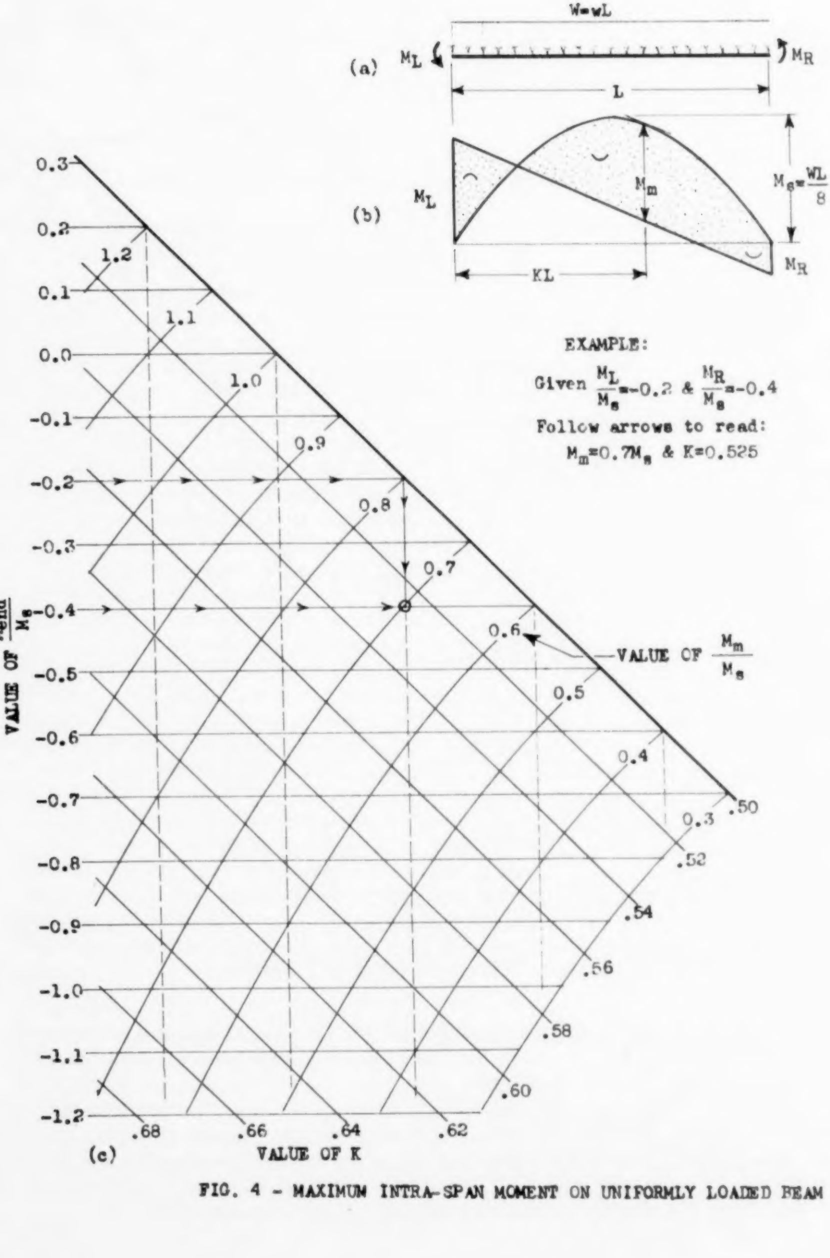


FIG. 4 - MAXIMUM INTRA-SPAN MOMENT ON UNIFORMLY LOADED BEAM

computing the relative plastic angles, θ'_p. These relative plastic angles for the assumed moments are then calculated and tabulated, assuming that the maximum M' is equal to M_u. The computation is shown in the left column. The change in plastic angle, $\Delta\theta'_p$, from the computed θ'_p of the previous trial is then tabulated. Noting the location of the point of maximum moment in the zone of plasticity and using C_m from Fig. 3, the changes in fixed-end moment, ΔFEM, caused by this change in plastic angle are computed. The corresponding unbalanced moments are then distributed, using the coefficients tabulated above. The computed moments for this cycle, M_3, are the summations of the computed moments for the previous cycle and the change in moments caused by the change in assumed plastic angles in this cycle. The discrepancy for this trial, δM_3, is the computed moment minus the assumed moment.

The next trial is started by assuming $\Delta M'_4$ equal to some portion of δM_3, in this example 0.50, an arbitrary but usually workable assumption. If δM is opposite to and greater than $\Delta M'$ for that cycle, the cycle should be crossed out and repeated with a new $\Delta M'$ being assumed at some lesser value.

"Elasti-plastic trial # 1" is of course somewhat modified since no previous plastic angle has, at that point, been incorporated into the computation. The assumed moments can safely be taken as the elastic solution, or, a designer who wishes to exercise his judgment to save computation may assume an initial set of reasonable increments of moment to modify the elastic solution.

An exact elasti-plastic solution is accomplished when all δM's are zero. However, observation of $\Delta M'$ in successive trials soon makes the final moments, within the desired accuracy, obvious so that a complete convergence is not necessary.

A check on the computations of all preceding elasti-plastic incremental trials may be made at any point by comparing the sum of the elastic solution moments and moment changes caused by all plastic angles, to the assumed moments for that cycle.

For this example the ratios of the strengths predicted by the elastic and rigid-plastic theories to the elasti-plastic strength are 0.80 and 1.03, respectively. To illustrate the variation which can be expected in these figures from structure to structure, if a fourth span, DE, loaded with 1 k/ft, were added to the structure of this example (producing symmetry about C), the corresponding ratios would be 0.78 and 1.19.

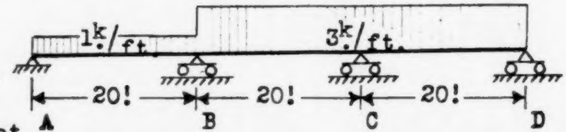
Example 2 (Fig. 6), Three-Span Continuous Frame

As a further illustration of the procedure, consider the reinforced concrete frame shown in Fig. 6, for which the width of all members is constant and for which the column depth is equal to 2/3 of the beam depth. The M - ϕ relationship for the beam is defined by k_1 equal to 7.55, k_2 negligible and M_u/M_e equal to 1.23. (These are the values given by the tentative relationships of reference (13) for 2.2% steel at plastic regions, a concrete strength of 4 ksi, a steel yield strength of 40 ksi, and a loading time of 4 hours.)

For the same materials and percentage of tensile steel, the values of k_1, k_2 and M_u/M_e are assumed to remain constant with variation in depth of beams. The introduction of axial load would probably affect the values of these factors; however, for the example this effect will be assumed negligible and the values listed above for the beams will also be used for the columns. On the other hand, M_u varies with the square of the depth, and the elastic stiffness

M-ϕ Constant, $k_1=0$,
$k_2=12$ & $M_u/M_e=1.05$

LOADED STRUCTURE →



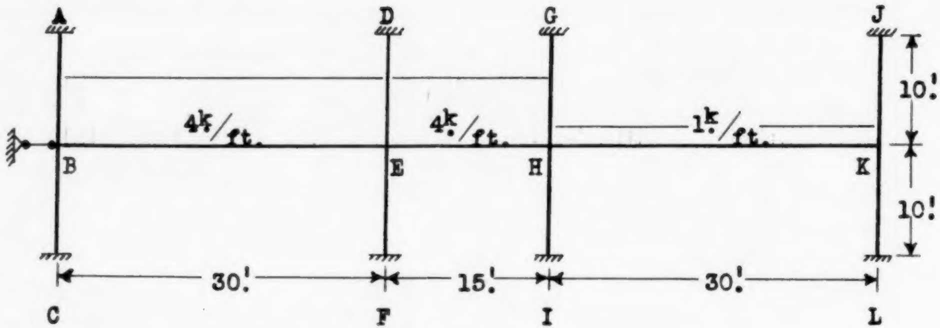
Geometric sign convention, values in kip-feet.

Remarks	Operation	Span AB	Joint B		Span BC	Joint C		Span CD
			BA	BC		CB	CD	
Simply-supported data	Shears & Moments	⌣ 50.0	10.0	30.0	⌣ 150.0	30.0	30.0	⌣ 150.0
Coef., distribution of	@ Joint B		⌒ 0.467	⌣ 0.533		⌒ 0.134	⌒ 0.134	
unit clockwise moment	@ Joint C		⌣ 0.134	⌣ 0.134		⌒ 0.533	⌣ 0.467	
	FEM		⌒ 50.0	⌒ 100.0		⌒ 100.0	⌒ 150.0	
Elastic Analysis	MD @ B		⌒ 23.3	⌣ 26.7		⌒ 6.7	⌒ 6.7	
	MD @ C		⌣ 6.7	⌣ 6.7		⌒ 26.7	⌣ 23.3	
	ELASTIC SOLUTION	⌣ 23.	⌒ 66.6	⌒ 66.6	⌣ 59.	⌒ 133.4	⌒ 133.4	⌣ 91.
Elas-plas. trial #1	V(elas. sol.)					33.3	36.7	
$M_e=133.4/1.05=127.0$	θ_p'(elas. sol.)					∧ 291.	∧ 264.	
θ_p'@C: $\frac{12\text{x}127.0\text{x}6.4}{V}$	Δ FEM(θ_p' @ C)			⌒ 29.1		⌣ 58.2	⌣ 39.6	
	MD @ B		⌒ 13.6	⌣ 15.5		⌒ 3.9	⌒ 3.9	
$M_1=$(MD+ΔFEM+el.sol.)	MD @ C		⌣ 2.5	⌣ 2.5		⌒ 10.0	⌣ 8.8	
$\delta M_1=M_1$-el.sol.	M_1		⌒ 77.7	⌒ 77.7		⌒ 89.1	⌒ 89.1	
	δM_1		⌒ 11.1			⌣ 44.3		
Elas-plas. trial #2	$\Delta M_2'$(say 0.5δM_1)		⌒ 5.6			⌣ 22.2		
$M_e=111.2/1.05=105.9$	M_2'($\Delta M_2'$+el.sol.)		⌒ 72.2	⌒ 72.2		⌒ 111.2	⌒ 111.2	
θ_p'@C: $\frac{12\text{x}105.9\text{x}5.3}{V}$	V					32.0	35.6	
	θ_p'					∧ 210.	∧ 189.	
$M_2=$(MD+ΔFEM+M_1)	$\Delta\theta_p'$					∨ 81.	∨ 75.	
$\delta M_2=M_2-M_2'$	ΔFEM($\Delta\theta_p'$ @ C)			⌣ 8.1		⌒ 16.2	⌒ 11.3	

	MD @ B		⌣ 3.8	⌢ 4.3		⌣ 1.1	⌣ 1.1	
	MD @ C		⌢ 0.7	⌢ 0.7		⌣ 2.6	⌢ 2.3	
	M_2		⌢ 74.6	⌢ 74.6		⌢ 101.6	⌢ 101.6	
	δM_2		⌢ 2.4			⌣ 9.6		
<u>Elas-plas. trial #3</u>	$\Delta M_3'$(say $0.5\delta M_2$)		⌢ 1.2			⌣ 4.8		
M_e=106.4/1.05=101.3	$M_3'(\Delta M_3'+M_2')$		⌢ 73.4	⌢ 73.4		⌢ 106.4	⌢ 106.4	⌣ 101.1
θ_p'@C: $\frac{12\text{x}101.3\text{x}5.1}{V}$	V					31.7	35.3	
	θ_p'					∧ 194.	∧ 174.	
M_3=(MD+ΔFEM+M_2)	$\Delta\theta_p'$					∨ 16.	∨ 15.	
$\delta M_3=M_3-M_3'$	ΔFEM($\Delta\theta_p'$ @ C)			⌣ 1.6		⌢ 3.2	⌢ 2.2	
	MD @ B		⌣ 0.7	⌢ 0.9		⌣ 0.2	⌣ 0.2	
	MD @ C		⌢ 0.1	⌢ 0.1		⌣ 0.5	⌢ 0.5	
	M_3		⌢ 74.0	⌢ 74.0		⌢ 104.1	⌢ 104.1	
	δM_3		⌢ 0.6			⌣ 2.3		
<u>Elas-plas. trial #4</u>	$\Delta M_4'$(say $0.5\delta M_3$)		⌢ 0.3			⌣ 1.2		
M_e=105.2/1.05=100.2	$M_4'(\Delta M_4'+M_3')$		⌢ 73.7	⌢ 73.7		⌢ 105.2	⌢ 105.2	⌣ 101.7
θ_p'@C: $\frac{12\text{x}100.2\text{x}5.0}{V}$	V					31.6	35.3	
	θ_p'					∧ 191.	∧ 171.	∨ 4400.
	$\Delta\theta_p'$					∨ 3.	∨ 3.	∨ 4400.
θ_p'onCD:2.83x12x	ΔFEM($\Delta\theta_p'$ @ C)			⌣ 0.3		⌢ 0.6	⌢ 0.4	
x100.2$(\frac{5.0}{3.0})^{\frac{1}{2}}$	ΔFEM($\Delta\theta_p'$ on CD)						⌢ 273.	
M_4=(MD+ΔFEM+M_3)	MD @ B		⌣ 0.1	⌢ 0.2				
$\delta M_4=M_4-M_4'$	MD @ C		⌣ 36.6	⌣ 36.6		⌢ 145.4	⌣ 127.4	
	M_4		⌢ 37.3	⌢ 37.3		⌢ 250.1	⌢ 250.1	
	δM_4		⌣ 36.4			⌢ 144.9		
Divergence is result of θ_p' forming on CD; obvious ELASTI-PLASTIC SOLUTION		⌣ 21.	⌢ 74.	⌢ 74.	⌣ 60.	⌢ 106.	⌢ 106.	⌣ 101.

FIG. 5 - DESIGN OF THREE-SPAN CONTINUOUS BEAM

Beams:
  M-ϕ constant, $k_1=7.55$,
  $k_2=0$ & $M_u/M_e=1.23$
Columns:
  Depths are 2/3 beam depths.
  M-ϕ as for beams above.



LOADED STRUCTURE

Geometric sign convention, values in kip-feet.

Operation	Joint B		Span	Joint E			Span	Joint H			Span	Joint K	
	*BC	BE	BE	EB	*EF	EH	EH	HE	*HI	HK	HK	KH	*KL
Pin-ended V & M	- -	60.0	450	60.0	- -	30.0	112½	30.0	- -	15.0	112½	15.0	- -
MD coef. @ B ↷	(0.658	⌣ 0.342		⌒ 0.142	) 0.071	⌒ 0.071		⌣ 0.023	(0.015	⌣ 0.008		⌒ 0.003	) 0.003
MD coef. @ E ↷	) 0.071	⌣ 0.071		⌒ 0.203	(0.399	⌣ 0.398		⌒ 0.127	) 0.084	⌒ 0.043		⌣ 0.015	(0.015
MD coef. @ H ↷	(0.015	⌒ 0.015		⌣ 0.043	) 0.084	⌣ 0.127		⌒ 0.398	(0.399	⌣ 0.203		⌒ 0.071	) 0.071
MD coef. @ K ↷	) 0.003	⌣ 0.003		⌒ 0.008	(0.015	⌒ 0.023		⌣ 0.071	) 0.071	⌣ 0.142		⌒ 0.342	(0.658
FEM		⌒ 300.0		⌒ 300.0		⌒ 75.0		⌒ 75.0		⌒ 75.0		⌒ 75.0	

MD ⌒300 @ B	(197.5	⌣102.5		⌒ 42.5	) 21.3	⌒ 21.3	⌣ 6.9	(4.5	⌣ 2.4	⌒ 0.9	) 0.9
MD ⌒225 @ E	(16.0	⌒ 16.0		⌣ 45.7	) 89.8	⌒ 89.5	⌣ 28.6	(18.9	⌣ 9.7	⌒ 3.4	) 3.4
MD ⌒ 75 @ K	(0.2	⌒ 0.2		⌣ 0.6	) 1.1	⌣ 1.7	⌒ 5.3	(5.3	⌒ 10.6	⌣ 25.6	) 49.3
ELASTIC SOLUTION	(213.7	⌒ 213.7	⌣ 196	⌒ 296.2	) 112.2	⌒ 184.1	⌒ 44.8	(28.7	⌒ 73.5	⌒ 53.7	) 53.6

Elasti-plastic trial #1

$M_{eb} = 296.2/1.23 = 240.8$

$M_{ec} = 240.8(8/9) = 214.0$

$\theta_p'(EB)$ @ E:

$$\frac{7.55x(55.4)^2}{2 \times 62.8} = 185$$

$M_1 = (MD + \Delta\, FEM + elas.sol.)$

$\delta M_1 = M_1 - elas.sol.$

Elasti-plastic trial #2

$M_{eb} = 283.6/1.23 = 230.6$

$M_{ec} = 230.6(8/9) = 205.0$

$\theta_p'(BC)$ @ B:

$$\frac{7.55x(13.1)^2}{2 \times 0.6 \times 21.8} = 50$$

$\theta_p'(EB)$ @ E:

$$\frac{7.55x(53.0)^2}{2 \times 62.2} = 170$$

$M_2 = (MD + \Delta\, FEM + M_1)$

$\delta M_2 = M_2 - M_2'$

Elasti-plastic trial #3

$M_{eb} = 282.8/1.23 = 230.0$

$M_{ec} = 230.0(8/9) = 204.4$

$\theta_p'(BC)$ @ B:

$$\frac{7.55x(9.9)^2}{2 \times 0.6 \times 21.4} = 29$$

$\theta_p'(EB)$ @ E:

$$\frac{7.55x(52.8)^2}{2 \times 62.3} = 169$$

$M_3 = (MD + \Delta\, FEM + M_2)$

$\delta M_3 = M_3 - M_3'$

FIG. 6 (a) - DESIGN OF THREE-SPAN CONTINUOUS FRAME

Operation	Joint B *BC	Joint B BE	Span BE	Joint E EB	Joint E *EF	Joint E EH	Span EH	Joint H HE	Joint H *HI	Joint H HK	Span HK	Joint K KH	Joint K *KL
TRIAL #1													
V (elas. sol.)	21.4	57.2		62.8									
θ_p'(elas. sol.)				∧185.									
Δ FEM(θ_p' @ E)		⌒12.3		⌒24.7									
MD @ B	(8.1	⌣4.2		⌒1.7	)0.9	⌒0.9		⌣0.3	(0.2	⌣0.1			
MD @ E	)1.8	⌣1.8		⌒5.0	(9.9	⌣9.8		⌒3.1	)2.1	⌒1.1		⌣0.4	(0.4
M_1	(220.0	⌒220.0		⌒278.2	)103.2	⌒175.2		⌒47.6	(26.8	⌒74.5		⌒53.3	)53.2
δM_1	(6.3	⌒6.3		⌣18.0	(9.0	⌣8.9		⌒2.8	)1.9	⌒1.0		⌣0.4	(0.4
TRIAL #2													
$\Delta M_2'$(say $0.7\delta M_1$)	(4.4	⌒4.4		⌣12.6	(6.3	⌣6.2		⌒2.0	)1.3	⌒0.7		⌣0.3	(0.3
M_2'($\Delta M_2'$+elas.sol.)	(218.1	⌒218.1	⌣200	⌒283.6	)105.9	⌒177.9		⌒46.8	(27.4	⌒74.2		⌒53.4	)53.3
V	21.8	57.8		62.2									
θ_p'	< 50.			∧170.									
$\Delta\theta_p'$	< 50.			∨ 15.									
Δ FEM($\Delta\theta_p'$ @ B)	)20.0												
Δ FEM($\Delta\theta_p'$ @ E)		⌣1.0		⌒2.0									
MD @ B	(12.5	⌣6.5		⌒2.7	)1.3	⌒1.3		⌣0.4	(0.3	⌣0.2		⌒0.1	)0.1
MD @ E	(0.1	⌒0.1		⌣0.4	)0.8	⌒0.8		⌣0.3	(0.2	⌣0.1			

M_2	(212.6	⌒ 212.6		⌒ 282.5	) 105.3	⌒ 177.3		⌒ 46.9	(27.3	⌒ 74.2		⌒ 53.4	) 53.3
δM_2	) 5.5	⌣ 5.5		⌣ 1.1	(0.6	⌣ 0.6		⌒ 0.1	) 0.1				
TRIAL #3													
$\Delta M_3'$(say $0.7\delta M_2$)	) 3.8	⌣ 3.8		⌣ 0.8	(0.4	⌣ 0.4		⌒ 0.1	) 0.1				
M_3' ($\Delta M_3' + M_2'$)	(214.3	⌒ 214.3	⌣ 202	⌒ 282.8	) 105.5	⌒ 177.5		⌒ 46.9	(27.3	⌒ 74.2		⌒ 53.4	) 53.3
V	21.4	57.7		62.3									
θ_p'	< 29.			^ 169.									
$\Delta\theta_p'$	> 21.			v 1.									
Δ FEM ($\Delta\theta_p'$ @ B)	(8.4												
MD @ B	) 5.5	⌒ 2.9		⌣ 1.2	(0.6	⌣ 0.6		⌒ 0.2	) 0.1	⌒ 0.1			
M_3	(215.5	⌒ 215.5		⌒ 281.3	) 104.7	⌒ 176.7		⌒ 47.1	(27.2	⌒ 74.3		⌒ 53.4	) 53.3
δM_3	(1.2	⌒ 1.2		⌣ 1.5	(0.8	⌣ 0.8		⌒ 0.2	) 0.1	⌒ 0.1			
ELASTI-PLASTIC SOLUTION	(*215.	⌒ 215.	⌣ 202	⌒ 282.	) *105.	⌒ 177.		⌒ 47.	(*27.	⌒ 74.		⌒ 53.	) *53.

*Total moment; each column one-half.

FIG. 6 (b) - DESIGN OF THREE-SPAN CONTINUOUS FRAME

M_e/ϕ_e or EI, varies as the cube of the depth, other factors remaining equal. Hence, if column depth equals 2/3 of beam depth, the combined ultimate strength of the two columns at each joint will be 8/9 of the strength of a beam, and the elastic stiffness of two columns will be 16/27, or approximately 0.6 that of a beam. That is:

$$(M_u)_c = \frac{8}{9}(M_u)_b$$

$$\left(\frac{M_e}{\phi_e}\right)_c = 0.6\left(\frac{M_e}{\phi_e}\right)_b$$

where subscripts c and b refer to column and beam, respectively. For convenience, in the computations of Fig. 6 the stiffnesses of the two columns at each joint have been combined, as well as the moments. It follows that the relative plastic angles formed in the beams and columns respectively will be:

$$(\theta_p')_b = \frac{7.55(M_m - M_{eb})^2}{2V}$$

$$(\theta_p')_c = \frac{7.55(M_m - M_{ec})^2}{(0.6)2V}$$

The computations for each trial parallel the computations for each trial of Example 1, to which reference may be made for explanation.

For this example the ratios of strengths predicted by the elastic method and the rigid-plastic method to elasti-plastic-calculated strength are 0.95 and 1.22, respectively.

Example 3 (Fig. 7), Frame with Sway

With a slight extension, the above method may be used for the elasti-plastic solution of continuous frames whose joints are free to translate or sway with one or more degrees of freedom.

As a preliminary step, moments in the elastic structure from a unit force acting in the direction of a freedom and on the joints which are free must be determined by conventional moment distribution or an equivalent method. Then the effect of sway may be included in an elasti-plastic solution by introducing into each step of the solution the increments of these sway forces needed for equilibrium of the swaying joints along with the elastic moment-increments resulting from these sway forces.

Example 3, Fig. 7, illustrates in detail a solution involving sway, using a purely arbitrary M - ϕ relationship defined by $k_1 = 0$, $k_2 = 12$, and $M_u/M_e = 1.10$. The structure shown has one degree of sway freedom in that member CF may translate laterally. Therefore, as a preliminary, the elastic moments in the structure from a unit lateral load acting on CF are found by any conventional method and entered as shown, following the other distribution coefficients. Then, for the elastic solution, after distribution of moments with sway prevented, the resulting horizontal forces on CF are computed (in this

case the three column shears and the 4^k external force). The resultant of these forces (5.50^k) is applied as a sway force to produce equilibrium, and the moments caused by this resultant are added to the non-sway moments.

For the elasti-plastic solution, similar sway corrections are merely introduced into each trial so that no unbalanced sway forces are associated with the final calculated moments of that trial. For example, for trial 1 of Example 3, additional increments of moment resulting from occurrence of the assumed plastic angles in the non-swaying structure are calculated as in previous examples. Then, since the sway forces were previously balanced for all previously calculated moments (in this case the elastic solution moments), it is only necessary to balance the resultant sway forces accompanying these new moment increments.

Example 3 also illustrates the assumption of arbitrary trial moments at the beginning of trial 1 in an effort to hasten convergence. It is usually not difficult to assume initial moments for trial 1 which are closer to the elasti-plastic solution than the elastic solution moments, thus hastening convergence. Of course, it is not necessary that the sway forces accompanying these arbitrary moments be balanced.

For those structures with members for which $k_1 = 0$, such as Example 3, it soon becomes apparent to the designer that, in establishing those trial moments exceeding M_e for each step, the relationship between the plastic angle and $(M - M_e)$ is more nearly linear than the relationship between the plastic angle and M. For Example 3, corresponding adjustments have been made for the trial moments increments marked thus (*), thereby hastening convergence.

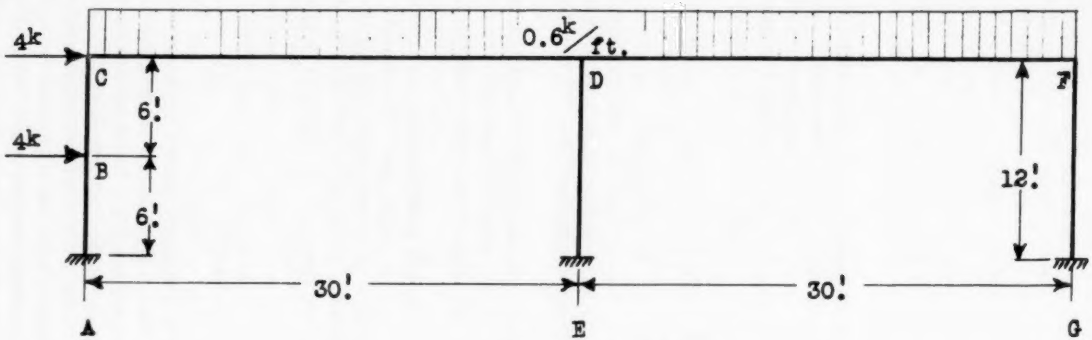
It is evident that, with experience in elasti-plastic calculations, the designer can expect to both improve his judgment and discover other techniques for improving the trial moments and thus reduce the number of trials necessary.

For example 3, the ratios of strength predicted by the elastic and rigid-plastic theories to the elasti-plastic strength are 0.74 and 1.24, respectively. It is obvious that these ratios, which have been given for all three examples, are not only functions of the geometry of the structure and the type of loading, but of k_1, k_2, and M_u/M_e; therefore they can be expected to vary over a considerable range.

Design

The method as demonstrated in the previous three examples has the limitation with respect to design of the traditional elastic analytical methods—the structure being designed must be known prior to the analysis. However, the method is more adaptable to design needs than shown in the examples in that the plastic portion of any moment-curvature relationships may be changed at the discretion of the designer from trial to trial of a solution. This flexibility is valuable to the designer of a constant-section steel frame who finds from the results of the first trial that he must use a section with an $M - \phi$ relationship appreciably different from that initially assumed. This flexibility is even more valuable to the designer of a reinforced concrete frame who, by changing placement of reinforcement from trial to trial, can change $M - \phi$ relationships and hence the distribution of moments and even the location of plastic regions at his discretion. Yu(14) and Baker(4) have investigated some design potentialities of such planned redistributions of moments.

M-ϕ constant, $k_1=0$,
$k_2=12$ & $M_u/M_e=1.10$



LOADED STRUCTURE

Geometric sign convention, values in kip-feet.

Operation	CA			CD			DE		DF			FG	
	A	AC	C	C	CD	D	D	E	D	DF	F	F	G
Pin-ended V & M	2	⟩12	2	9	⌣67½	9			9	⌣67½	9		
MD coef. @ C ↶	⟩ 0.363		⟨ 0.726	⌣ 0.274		⌢ 0.112	⟩ 0.081	⟨ 0.040	⌢ 0.031		⌣ 0.012	⟨ 0.012	⟩ 0.006
MD coef. @ D ↶	⟨ 0.041		⟩ 0.082	⌣ 0.082		⌢ 0.213	⟨ 0.574	⟩ 0.287	⌣ 0.213		⌢ 0.082	⟩ 0.082	⟨ 0.041
MD coef. @ F ↶	⟩ 0.006		⟨ 0.012	⌢ 0.012		⌣ 0.031	⟩ 0.081	⟨ 0.040	⌣ 0.112		⌢ 0.274	⟨ 0.726	⟩ 0.363
M (CF→1 sway)	⟨ 2.36		⟩ 1.21	⌣ 1.21		⌢ 1.04	⟩ 2.07	⟨ 2.79	⌣ 1.04		⌢ 1.21	⟩ 1.21	⟨ 2.36

FEM	< 6.0		< 6.0	∩ 45.0		∩ 45.0			∩ 45.0		∩ 45.0		
MD @ C(39.0 ↶)	> 14.1		< 28.3	∪ 10.7		∩ 4.4	> 3.2	< 1.6	∩ 1.2		∪ 0.5	< 0.5	> 0.2
MD @ F(45.0 ↷)	< 0.3		> 0.5	∪ 0.5		∩ 1.4	< 3.6	> 1.8	∩ 5.0		∪ 12.3	> 32.7	< 16.3
Partial Σ	> 7.9		< 33.8				< 0.4	> 0.2				> 32.2	< 16.1
V			→ 5.47	→ 4.00			→ 0.05					← 4.02	
M (CF → 5.50)	< 13.0		> 6.7	∪ 6.7		∩ 5.7	> 11.4	< 15.3	∪ 5.7		∩ 6.7	> 6.7	< 13.0
ELASTIC SOLUTION	< 5.2	< 4.1	< 27.1	∩ 27.1	∪ 26	∩ 56.5	> 11.0	< 15.1	∩ 45.5	∪ 26	∩ 38.9	> 38.9	< 29.1

Elasti-plastic trial #1

$M_e = 45.0/1.10 = 40.9$

$\theta_p'(CD)$ @ D:
$(12 \times 40.9 \times 4.1)/9.50 = 212.$

$\theta_p'(DF)$ @ F:
$(12 \times 40.9 \times 1.1)/9.07 = 59.$

$\theta_p'(FG)$ @ F:
$(12 \times 40.9 \times 1.1)/6.33 = 85.$

Elasti-plastic trial #2

$M_e = 43.6/1.10 = 39.6$

$\theta_p'(CD)$ @ D:
$(12 \times 39.6 \times 4.0)/9.47 = 201.$

$\theta_p'(DF)$ @ D & F:
$(12 \times 39.6 \times 0.6)/9.00 = 32.$

$\theta_p'(FG)$ @ F:
$(12 \times 39.6 \times 0.6)/6.21 = 46.$

Elasti-plastic trial #3

$M_e = 42.2/1.10 = 38.4$

$\theta_p'(CD)$ @ D:
$(12 \times 38.4 \times 3.8)/9.41 = 186.$

$\theta_p'(DF)$ @ D:
$(12 \times 38.4 \times 0.4)/8.99 = 21.$

$\theta_p'(DF)$ @ F:
$(12 \times 38.4 \times 0.5)/9.01 = 26.$

$\theta_p'(FG)$ @ F:
$(12 \times 38.4 \times 0.5)/6.11 = 38.$

FIG. 7 (a) - DESIGN OF FRAME FREE TO SWAY

Operation	CA			CD			DE		DF			FG	
	A	AC	C	C	CD	D	D	E	D	DF	F	F	G
M'_1 (Assumed)	⊂ 5.0		⊂ 30.0	∩ 30.0		∩ 45.0	⊃ 5.0	⊂ 12.0	∩ 40.0		∩ 42.0	⊃ 42.0	⊂ 34.0
V						9.50					9.07	6.33	
θ'_p						∧ 212.					∧ 59.	> 85.	
Δ FEM				∩ 14.1		∪ 28.2			∩ 3.9		∪ 7.9	⊂ 28.3	⊃ 14.2
MD @ C(14.1↷)	⊃ 5.1		⊂ 10.2	∪ 3.9		∩ 1.6	⊃ 1.1	⊂ 0.6	∩ 0.4		∪ 0.2	⊂ 0.2	⊃ 0.1
MD @ D(32.0↷)	⊂ 1.3		⊃ 2.6	∪ 2.6		∩ 6.8	⊂ 18.4	⊃ 9.2	∪ 6.8		∩ 2.6	⊃ 2.6	⊂ 1.3
MD @ F(20.4↶)	⊂ 0.1		⊃ 0.2	∪ 0.2		∩ 0.6	⊂ 1.7	⊃ 0.8	∩ 2.3		∪ 5.6	⊃ 14.8	⊂ 7.4
Partial Σ	⊃ 3.7		⊂ 7.4				⊂ 19.0	⊃ 9.4				⊂ 11.1	⊃ 5.6
ΔV(from θ'_p)			→0.93				→2.37					→1.35	
M (CF→4.65)	⊂ 11.0		⊃ 5.6	∪ 5.6		∩ 4.8	⊃ 9.6	⊂ 13.0	∪ 4.8		∩ 5.6	⊃ 5.6	⊂ 11.0
M_1	⊂ 12.5		⊂ 28.9	∩ 28.9		∩ 42.1	⊃ 1.6	⊂ 18.7	∩ 40.5		∩ 33.4	⊃ 33.4	⊂ 34.5
δM_1	⊂ 7.5		⊃ 1.1	∪ 1.1		∪ 2.9	⊂ 3.4	⊂ 6.7	∩ 0.5		∪ 8.6	⊂ 8.6	⊂ 0.5
$\Delta M'_2$(say $0.5\delta M_1$)	⊂ 3.8		⊃ 0.6	∪ 0.6		∪ 1.4	⊂ 1.7	⊂ 3.4	∩ 0.2		∪ *1.8	⊂ *1.8	⊂ 0.3
M'_2	⊂ 8.8		⊂ 29.4	∩ 29.4		∩ 43.6	⊃ 3.3	⊂ 15.4	∩ 40.2		∩ 40.2	⊃ 40.2	⊂ 34.3
V						9.47			9.00		9.00	6.21	
θ'_p						∧ 201.			∧ 32.		∧ 32.	> 46.	
$\Delta\theta'_p$						∨ 11.			∧ 32.		∨ 27.	< 39.	
Δ FEM($\Delta\theta'_p$ @ D)				∪ 0.7		∩ 1.5			∪ 4.3		∩ 2.1		
Δ FEM($\Delta\theta'_p$ @ F)									∪ 1.8		∩ 3.6	⊃ 13.0	⊂ 6.5
MD @ C(0.7↶)	⊂ 0.3		⊃ 0.5	∩ 0.2		∪ 0.1	⊂ 0.1						
MD @ D(5.8↶)	⊃ 0.2		⊂ 0.5	∩ 0.5		∪ 1.2	⊃ 3.3	⊂ 1.7	∩ 1.2		∪ 0.5	⊂ 0.5	⊃ 0.2
MD @ F(7.3↷)			⊂ 0.1	∩ 0.1		∪ 0.2	⊃ 0.6	⊂ 0.3	∪ 0.8		∩ 2.0	⊂ 5.3	⊃ 2.6
Partial Σ	⊂ 0.1		⊂ 0.1				⊃ 3.8	⊂ 2.0				⊃ 7.2	⊂ 3.7

ΔV(from $\Delta\theta_p'$)							←0.48					←0.91	
M (CF←1.39)	) 3.3		(1.7	∩ 1.7		∪ 1.4	(2.9	) 3.9	∩ 1.4		∪ 1.7	(1.7	) 3.3
M_2	(9.3		(30.7	∩ 30.7		∩ 40.7	) 2.5	(16.8	∩ 36.2		∩ 38.9	) 38.9	(34.9
δM_2	(0.5		(1.3	∩ 1.3		∪ 2.9	(0.8	(1.4	∪ 4.0		∪ 1.3	(1.3	(0.6
$\Delta M_3'$(say 0.5δM_2)	(0.3		(0.6	∩ 0.6		∪ 1.4	(0.4	(0.7	∪ *1.4		∪ *1.3	(*1.3	(0.3
M_3'	(9.1		(30.0	∩ 30.0		∩ 42.2	) 2.9	(16.1	∩ 38.3		∩ 38.9	) 38.9	(34.6
V						9.41			8.99		9.01	6.11	
θ_p'						∧ 186.			∧ 21.		∧ 26.	> 38.	
$\Delta\theta_p'$						∨ 15.			∨ 15.		∨ 6.	< 8.	
ΔFEM($\Delta\theta_p'$ @ D)				∪ 1.0		∩ 2.0			∩ 2.0		∪ 1.0		
ΔFEM($\Delta\theta_p'$ @ F)									∪ 0.4		∩ 0.8	) 2.7	(1.3
MD @ C(1.0↶)	(0.4		) 0.7	∩ 0.3		∪ 0.1	(0.1						
MD @ D(0.4↶)						∪ 0.1	) 0.2	(0.1	∩ 0.1				
MD @ F(2.9↷)						∪ 0.1	) 0.2	(0.1	∪ 0.3		∩ 0.8	(2.1	) 1.1
Partial Σ	(0.4		) 0.7				) 0.3	(0.2				) 0.6	(0.2
ΔV(from $\Delta\theta_p'$)			←0.09				←0.04					←0.07	
M (CF←0.31)	) 0.5		(0.2	∩ 0.2		∪ 0.2	(0.4	) 0.6	∩ 0.2		∪ 0.2	(0.2	) 0.5
M_3	(9.2		(30.2	∩ 30.2		∩ 42.2	) 2.4	(16.4	∩ 37.8		∩ 39.3	) 39.3	(34.6
δM_3	(0.1		(0.2	∩ 0.2			(0.5	(0.3	∪ 1.0		∩ 0.4	) 0.4	
ELASTI-PLASTIC SOLUTION	(9.2		(30.	∩ 30.		∩ 42.	) 2.7	(16.	∩ 39.		∩ 39.	) 39.	(35.

FIG. 7 (b) - DESIGN OF FRAME FREE TO SWAY

Deflections

Elasti-plastic deflections of continuous beams and frames may be calculated by the same techniques used to calculate elastic deflections of such structures. That is, after moments and plastic angles have been determined by an elasti-plastic solution, deflections may be calculated by any usual method provided these plastic angles are included. For the moment area method, localized areas equal to each plastic angle, θ_p (or θ'_p/EI), should be added to the M/EI diagram, and for the conjugate beam method, concentrated loads equal to each plastic angle are added to the conjugate beam, following which deflections may be calculated by the usual techniques of the respective method.

Evaluation of Method

The method of this paper is valuable in that, unlike the rigid-plastic methods, it allows determination of ultimate strength as limited by an ultimate deformation as well as ultimate moment. That is, by mere adjustment of coefficients which express the magnitude and strain-hardening characteristics of the plastic deformation, it may be used for ultimate load determination for structures of any material ranging from the most brittle to the most ductile and with any number of plastic regions. Also, it provides data from which elasti-plastic deflections may be readily calculated. However, all other limitations characteristic of limit design methods apply to this method. These limitations have been discussed in a previous paper.[2]

REFERENCES

1. "Theory of Inelastic Bending with Reference to Limit Design," Hrennikoff, A., Transactions ASCE, Vol. 113 (1948), pp. 213-268.

2. "Elasti-plastic Design of Single-span Beams and Frames," Sawyer, H. A. Jr., Proceedings ASCE, Separate 851, Vol. 81 (December 1955); No. ST4, Vol. 82 (July 1956), pp. 1024-12, 13, and No. ST1, Vol. 83 (January 1957), pp. 1156-3 - 6.

3. "A Method for the Analysis of Frames Subjected to Inelastic Deformation into the Range of Strain-hardening," Ang, A. and Massard, J. M., University of Illinois, Report of Project 1080, Contract AF33 (616)-170, Air Force Special Weapons Center, Kirtland Air Force Base, New Mexico, November 1956.

4. Ultimate Load Theory Applied to the Design of Reinforced and Prestressed Concrete Frames, Baker, A. L. L., Concrete Publications Ltd., London, 1956.

5. "The Ultimate Strength and Deformation of Plastic Hinges in Reinforced Concrete Frameworks," Chan, W. W. L., Magazine of Concrete Research, Vol. 7, No. 21 (November 1955), pp. 121-132.

6. "A Study of the Resistance of Model Frames to Dynamic Lateral Load," Mayerjak, R. J., Civil Engineering Studies, Structural Research Series No. 108, University of Illinois, August 1955.

7. "Shear Deflection of Wide Flange Steel Beams in the Plastic Range," Hall, W. J. and Newmark, N. M., Proceedings ASCE, Separate 814, Vol. 81 (October 1955).

8. "The Plastic Behavior of Structural Members and Frames," Driscoll, G. C. and Beedle, L. S., Welding Journal, Vol. 36, No. 6 (June 1957), pp. 275s-286s.

9. "An Investigation of the Load-Deformation Characteristics of Reinforced Concrete Beams up to the Point of Failure," Gaston, J. R., Siess, C. P. and Newmark, N. M., Civil Engineering Studies, Structural Research Series No. 40, University of Illinois, December 1952.

10. "Load-Deformation Characteristics of Simulated Beam Column Connections in Reinforced Concrete," McCollister, H. M., Siess, C. P. and Newmark, N. M., Civil Engineering Studies, Structural Research Series No. 76, University of Illinois, June 1954.

11. "Analysis of Continuous Frames by Distributing Fixed End Moments," Cross, H., Transactions ASCE, Vol. 96 (1932), pp. 1-156.

12. "Analysis of Continuous Structures by Joint Rotation," Looney, C. T. G., Proceedings ASCE, Separate 679, Vol. 81 (May 1955).

13. "The Behavior of Under-Reinforced Concrete Beams under Long-Term Loads," Sawyer, H. A., Stephens, J. E. and Alling, E. S., Bulletin, University of Connecticut Engineering Experiment Station, No. 12 (September 1956).

14. An Analytical Investigation of the Ultimate Load Design of Reinforced Concrete Framed Structures, Yu, C. W., Thesis for degree Ph.D., University of London, 1954.

Journal of the

STRUCTURAL DIVISION

Proceedings of the American Society of Civil Engineers

CONTENTS

DISCUSSION
(Proc. Paper 1882)

(over)

*There will be no closure.

TENTATIVE RECOMMENDATIONS FOR PRESTRESSED CONCRETE: REPORT OF THE JOINT ACI-ASCE COMMITTEE ON PRESTRESSED REINFORCED CONCRETE[a]

Closure by the Committee

COMMITTEE'S CLOSURE.—

INTRODUCTION

This closure will deal with published discussions as well as unpublished comments and will include some general explanations where deemed desirable. Therefore, the best procedure is to arrange the closure in the sequence of sections in the report rather than by name of discussors.

All but one of the discussions appeared in both the Journal of the Structural Division of ASCE and the Journal of the American Concrete Institute. One discussion appeared only in the Journal of the Structural Division.

101-Objective

This report is a recommended practice, not a specification. It is presented for the guidance of professional engineers. Safety, performance, and economy will depend as much on the intelligence and integrity of engineers as on the degree to which these recommendations are followed. This is taken almost verbatim from Sec. 101. The brevity of the restatement makes the intent stand out in stronger relief. The re-emphasis seems particularly desirable because many of the comments seem to assume that the report is a code or specification.

A code is a legal document that may be incorporated, in whole or in part, in a specification that is legally binding on all parties involved. By contrast, these tentative recommendations express the viewpoints generally held by a majority of specialists in this particular field, and the advisory form of presentation was adopted as best suited to the assignment of the committee.

Attention is called to A. W. Coutris' thoughtful discussion of "Advisory Reports" as opposed to "Codes." He pleads for keeping design reports advisory rather than intransigent in form and spirit.

102-Scope

The recommendations relate mostly to beams, girders, and slabs. Tanks and pipes are specifically omitted for the reasons stated in Sec. 102.1. It has been suggested that piling should have been included. Since stress conditions

a. Proc. Paper 1519, January, 1958.

in piles differ appreciably from those in purely flexural members the same design recommendations do not apply to both types. When the report is revised, prestressed piling probably should be included in the scope.

L. D. Boswell and others suggest that a definition for "segmental elements" be added in Sec. 102.2. The committee proposes a definition as follows: A member may be cast integrally as a "single element" or comprised of a number of precast pieces, in which case it is referred to as a "segmental element."

104-Notation

Several years ago the committee prepared a report on notation which was published in the ACI Journal.* In the course of preparing the present report the original notation was modified and augmented as deemed desirable. P.W. Abeles states that "the symbols now published are greatly improved as compared with the first publication of notations." Nevertheless, he regrets that his efforts to obtain uniformity of symbols in the United States and Great Britain have been unsuccessful.

On previous occasions, Abeles proposed the adoption of an international notation for prestressed concrete. While recognizing the value of such a generally applicable notation, the committee felt it was in no position to assume the prerogative of speaking for all groups concerned with notation in the United States.

Abeles' suggestions about augmenting the definitions for $\underline{k}$ and $\underline{\delta}$ in Sec. 104.4 are helpful improvements.

201.2-Mode of failure

In reference to the wording in this section, G. F. Janssonius states that "most test beams have collapsed when ultimate compressive strength of the concrete was reached." In the United States it is common to state the case just as expressed in Sec. 201.2. In almost all test beams designed to fail in flexure, ultimate <u>collapse</u> manifests itself as crushing of concrete in the compression zone; but in under-reinforced sections, ultimate <u>strength</u> is said to be reached when the tensile reinforcement undergoes permanent strain in its plastic region.

201.3-Design theory

T. Y. Lin apparently misinterprets the meaning and intent of this section. It simply emphasizes that both performance under service conditions and strength under ultimate conditions should be investigated.

Comparing the design approach in these recommendations with that in two other reports, Lin states that a "basic disagreement exists." It is difficult to perceive any real justification for the alleged basic disagreement.

The only specific value Lin uses "as an example of the dangerous errors contained in these allowable stresses. . ." is in reference to $3\sqrt{f'_{ci}}$ in

*"Proposed Definitions and Notations for Prestressed Concrete," Journal of American Concrete Institute, Vol. 24, Part 2.

Sec 207.3.1 (b). This is an isolated stress condition of relatively minor significance. It does not represent the general, complex conditions of "strength and behavior of beams at transfer," to which he has reference.

204.3-Initial prestress plus dead load of member

This clause simply describes one of the significant loading stages that normally affects design. It was not intended, as Abeles infers, to explain how to compute stresses. It was thought desirable, however, to state that losses in prestress at that stage should not include long-range or sustained load effects.

Sec. 204.6-Cracking load

A brief commentary on the significance of cracking load seems desirable, especially since many unpublished comments have been received on this subject.

Recommendations related to cracking appear in Sections 204.6, 205.2, and 207.3.3. All these combined express the intent of the committee. Sec. 204.6 states that "complete freedom from cracking may or may not be necessary... Type and function of the structure, and type, frequency, and magnitude of live loads should be considered." In the same vein, it is stated in Sec. 205.2 that "formation of a crack under temporary overload may not be objectionable."

Previous recommendations and criteria for cracking frequently have been quite arbitrary and inflexible. The committee feels that the profession gradually has moved away from arbitrary edicts on this subject. At one stage of the deliberations, a proposal was discussed regarding complete omission of references to cracking loads and stresses. This was rejected, however, from the viewpoint that cracking and its various effects did remain significant for many types of structures, loadings, and exposures such as in girders subject to impact, repetitive loading and exposure to harmful atmospheric vapors.

205.3-Ultimate load factors

Ultimate load capacity expresses a basic characteristic of almost all concrete structures. For prestressed concrete, the significance of ultimate load capacity is further emphasized by the fact that stresses are not linearly proportional to external forces and moments throughout the entire load range.

Considerable work had already been done by others on the subject of ultimate load factors for buildings. The principal features of this work are abstracted in the Building Code Requirements for Reinforced Concrete (ACI 318-56). Ultimate load factors for buildings in that code were incorporated in this report.

Load factors for highway bridges were studied at considerable length before the final decision was made to adopt the factors of 1.5 for dead load and 2.5 for live load. Numerous combinations of factors other than these were also tried as bases for calculations of actual bridge structures.

As a result of the bridge studies, two opposing viewpoints developed. One group felt that a single expression such as 1.5D + 2.5L would suffice, whereas others preferred a "dual" expression that also contained a safeguard for

structures with very low ratios of live to dead load. The text in Sec. 205.3 represents a compromise in that one single expression is recommended, but with the qualification that "it may be desirable to modify or expand the load factor formulas to fit special conditions."

The recommendations related to load factors for highway bridges were formulated in cooperation with both federal and state bridge authorities.

207-Allowable steel and concrete stresses

The specific stress values in this section were chosen after thorough study of all pertinent data. During the last year before publication of the report, numerous comments were received and some modifications made in the allowable stresses. It is felt that the values published reflect the very best on which agreement could be obtained.

In reference to some of Lin's remarks about allowable stresses, it should be reaffirmed that the provisions in Sec. 207 are intended to be advisory rather than intransigent. Special circumstances may dictate a downward revision of certain values. Liberalization may be indicated in other instances where sufficient supporting data can be submitted, including analytical studies, test results, or performance records. In the latter case, the burden of proof should fall upon those who wish to deviate from the generally accepted values.

207.1-Prestressing steel

The recommendations define three allowable steel stresses:

$0.80\ f'_s$: for a short period of time prior to seating of anchorage.

$0.70\ f'_s$: immediately after seating but before losses.

$0.60\ f'_s$: effective steel stress after losses described in Sec. 208.

The first value was chosen because some steels have shown marked stress relaxation when stressed above $0.80f'_s$, even for relatively short periods of time. When the prestress force is transferred from the jack to the anchorage, some slip and displacement may occur in various parts of most anchorages. The loss in stresses during seating often is of appreciable magnitude. Further losses occur due to sources described in Sec. 208. The three values for allowable steel stresses are interrelated in such a way that it is impossible to discuss only one without reference to the other two.

C. J. Fox indicates that he prefers $0.80f'_s$ to $0.70f'_s$ but fails to state whether this is "before" or "after" seating of anchorage. If he means $0.80f'_s$ <u>before</u> seating, then there is no disagreement.

207.3.1-Temporary stresses

G. F. Janssonius asks why the recommendations allow temporary concrete stresses of $0.60f'_{ci}$ for pretensioned members and $0.55f'_{ci}$ for post-tensioned members.

Here, production had preceded design recommendations, and the stress of $0.60f'_{ci}$ had already been widely established in the pretensioning industry. No

ill effect had been reported in regard to strength and performance. Only camber proved difficult to control for certain building members.

It appeared desirable to be more conservative for post-tensioning. One reason is that the effect of holes left for post-tensioning reinforcement is frequently ignored, and designers are inclined to use gross rather than net section. Also, as cables are post-tensioned, one after the other, eccentric prestressing forces are introduced which may create temporary stress conditions not anticipated. Finally, control and inspection usually are not as good in the field as in a plant.

D. J. Oswald says his firm obtains 28-day strengths between 6,500 and 7,500 psi using Type II cement without additives. His suggestion is that the report should have included a minimum limit for concrete strength.

Prestressed concrete can be produced with satisfactory technical results using concrete strengths varying over a large range, even if it may not be economically advantageous to use extreme values. On this subject the committee had no access to pertinent technical information. Local conditions differ as to types and gradations of aggregate. Also, there may be differences in equipment and in mechanical means available for mixing, transporting and placing concrete. It was felt that in the choice of concrete strength, it is the design engineer's problem, and the final decision is up to him.

207.3.2-Stresses at design loads

Janssonius calls attention to the values for flexural tension in Sec. 207.3.2(b): $6\sqrt{f'_c}$ for pretensioned and $3\sqrt{f'_c}$ for post-tensioned, bonded elements. In his opinion, "so great a difference, if any, is not justified."

This is another instance in which the pretensioning industry for many years had followed a standard of production that had given satisfactory results. Pretensioning elements usually consist of a relatively large number of strands well distributed in and bonded to the concrete. A high degree of bond resistance ensures good crack control. Strictly from the viewpoint of bonding characteristics, the servicability and behavior in regard to crack control should be better in pretensioning than in post-tensioning.

Whereas there seemed to be good reason for allowing different stresses for flexural tension, the committee had insufficient technical information on which to base the specific values in the report. Consequently, a clause was added permitting certain variations from the values in Sec. 207.3.2(b).

207.3.3-Stress at cracking load

This clause states that when test data are not available the ultimate flexural tensile stress in psi may be assumed as:

$$f'_t = 7.5\sqrt{f'_c}$$

The particular wording was purposely chosen so as to make it plain that "test data" should be given more weight than the recommended allowable stress value, on which it was difficult to obtain agreement.

For, say, $f'_c = 6400$ psi, the value allowed for f'_t is 600 psi. Both Abeles and Bryan propose that this could be raised to 1,000 psi under certain

circumstances or in certain types of structures. The committee does not consider the recommended stress formula restrictive in an absolute sense. This problem may be resolved by referring to clauses in Sections 204.6 and 205.2 with a well supported argument in favor of deviation from the recommended value.

208.2.1-Friction loss in post-tensioned steel

Test results available on cable friction do not cover nearly all the variable conditions that are needed to develop a comprehensive set of accurate values for K and μ.

Values of K and μ in Sec. 208.2.1 are considered conservative for materials and construction procedures used in the United States. It is anticipated that the values will be revised and augmented as newer research data become available. In this respect, it is advisable to note A. W. Coutris' contribution to the subject of friction and to refer to some of the sources mentioned in his discussion.

Coutris believes the values given in the table are conservative for wires, but not conservative enough for bars. One difficulty that complicates the subject of friction loss is that materials and especially workmanship may vary widely. Accuracy in placing ducts plays an important part and is difficult to control in practice and evaluate in tests.

208.2.3-Shrinkage of concrete

Several discussers suggest that more specific information be included on shrinkage. The committee felt that the effect of shrinkage in prestressed members was not such as to justify the inclusion of a comprehensive treatment.

The choice of shrinkage factor usually has no effect upon ultimate strength. At cracking load, the flexural tensile strength of concrete is equally as important as shrinkage, or more so. In camber calculations, creep rather than shrinkage tends to be the predominant factor. Also it is impossible at present to give shrinkage data that will apply to lightweight concretes in general.

208.3.2-Method 2

The assumption regarding losses in steel stresses in Sec. 208.3.2 reads:

Pretensioning	35,000 psi
Post-tensioning	25,000 psi

W. J. Jurkovich states that "the pretensioning loss of 35,000 psi appears to be on the high side. The preliminary draft of the report presented a range of losses (30,000-40,000 and 20,000-30,000) which appeared reasonable." D. J. Oswald also believes the loss of 35,000 psi is conservative and suggests reducing it to 30,000 psi. On the other hand, P. W. Abeles suggests that Method 2 either be eliminated or the loss value increased.

Actually, the ultimate strength is not significantly affected by the magnitude of steel stress loss. For camber calculations, the suggested values may

be excessive. Making the distinction between 'strength' and 'camber' is important when applying stress losses in accordance with both methods in Sec. 208.3.

209.2-Ultimate flexural strength

The comments received on this section have been mainly discussions of the k-factors in equation (a). The symbols of k_2 and k_1k_3 were introduced originally as useful quantities in analytical studies on ultimate flexural compressive stresses in concrete. This committee incorporated the symbols in its report in order to give equation (a) a fully general form.

Research data showed that the general algebraic form of equation (a) could be simplified for the great majority of conditions by inserting 0.6 for k_2/k_1k_3. This gives slightly conservative results. The use of the ratio as a factor in the second member within the brackets in equation (a) makes illusory any high degree of refinement in the k-fractions for members and materials considered in this report.

209.2.1(b)-Ultimate flexural strength of flanged sections

A question has been asked about the definition of A_{sr}. In equation (b) the width b', should be the average width of that particular portion of the stem which is in compression at ultimate load.

Attention is called to the fact that the value of A_{sr} in equation (b) may become negative. This happens when the web width, b', is less than 15 per cent of the total width, b. For values of b' smaller than 0.15b it is suggested that equation (b) be disregarded and replaced by equation (a) in Sec. 209.2.1(a).

209.2.3(b)-Flanged sections

Correction: In the first term in the equation for M_u, change $\underline{b}$ to $\underline{b}'$.

210.2.2-Design of web reinforcement

It is stated in one unpublished discussion that "design by the conventional formula given with its factor of 1/2 may not be safe" because "the principal shearing stress increases far more rapidly than the increase in external load." The erroneous inference is that the equation in Sec. 210.2.2 is applicable to service loading. It is actually an ultimate strength equation as plainly shown by the notation given below the equation. The 1/2 is not related to a load factor.

The design recommended for web reinforcement is based on tests of limited scope, principally tests on simply supported beams with straight prestressing steel. The equation in Sec. 210.2.2 may not be applicable to continuous structures. For portions of simply supported beams where the steel is draped, the designer should consider using one of two limit values of $\underline{d}$. Choosing $\underline{d}$ as the depth to the center of gravity of steel may be too conservative at points of the beam where steel is draped. On the other hand, using the value of $\underline{d}$ at midspan throughout the beam may not be conservative at points where steel is draped.

The required value of A_V depends on many variables including prestress force, web thickness, amount of tensile reinforcement, and shear/moment ratio. There were not sufficient data to enable the committee to evaluate all such variables in algebraic form. The expression generally applicable to reinforced concrete was adopted and the factor of one-half added to make allowance for the beneficial effect of the prestress force with the precaution added that "it may not be conservative for very low prestress or where only a portion of the reinforcement is stressed." The form of the equation is not chosen solely because it is similar to that for reinforced concrete. Actually, the formula shows good correlation with results of many tests on typical prestressed concrete members.

210.2.4-Spacing of web reinforcement

It should be noted that spacings in this section are 'maximum' rather than 'optimum' values. In general, maximum values are not necessarily the best to be recommended but smaller values may often be more desirable to use.

213.3-Frictional losses

G. F. Janssonius correctly states that the clause in Sec. 213.3 is not clear. It reads that "frictional losses in continuous post-tentioned steel may be more significant than in simply supported members." Whether a structure is continuous or not has no intrinsic effect on frictional losses. The trouble is that the clause is abbreviated to the point where it gives the wrong inference. It was meant to express that since continuous structures are generally longer, and the post-tensioned elements may have reverse curvature, frictional losses tend to be greater in continuous than in simply supported members.

213.4-Ultimate strength

Coutris feels that "moment redistribution should be considered in design" in order to "predict the ultimate strength of . . . continuous beams. . . ," whereas Sec. 213.4 "recommends that moment redistribution not be considered in design at the present time."

Attention is called to Sec. 204.8 which reads in part: "In statically indeterminate structures, the load which causes moment in one section to reach its ultimate value may not be sufficient to cause failure of the structure because of moment redistribution. Since it is not always possible to predict that full redistribution will take place in accordance with limit design, it is suggested for the time being that moments be determined by elastic analysis."

On the subject of elastic versus limit design, the committee was cognizant of two pertinent points. Conventional reinforced concrete structures are still being designed by the elastic theory with no allowance being made for moment redistribution, and practical step-by-step procedures and design-aids are not available to be of sufficient help and guidance in limit design of continuous prestressed structures.

302.4-Strength

L. P. Marchant proposes that reference be made in Sec. 302.4 to "Recommended Practice for Evaluation of Compression Test Results of Field Concrete" (ACI 214-57) in addition to the reference now made to "Building Code Requirements for Reinforced Concrete (ACI 318-56). The procedure outlined in the report of ACI 214 is useful for plant operation, but the committee feels the procedure has not been in use long enough to prove it is equally practical for strength evaluation of concrete mixed on the site.

304-Prestressing steel

Before any official specifications existed in the United States, prestressing steels were manufactured and used extensively in structures. Changes and improvements were made in some instances by the manufacturers wherever desirable. This is one of several instances in which production preceded specifications. Since these steels have a long and successful performance record it is surely inadvisable, as Michael Chi suggests, arbitrarily to increase requirements for ductility to a point which none of the steels can reach.

A committee appointed by the American Society for Testing Materials is preparing specifications for prestressing steel. Since ASTM possesses authority and jurisdiction, this committee has established liaison with the ASTM group but is not otherwise responsible for specifications.

It is apparently desirable to reaffirm that Sec. 304 simply describes four types of prestressing steel now in common use. This description should not be interpreted as a specification or a testimonial. If types of prestressing steel other than those described are proposed to be added or substituted, requests for specifications should be directed to ASTM.

304.2.2-Shape of stress-strain curve

The definitions of minimum yield strength in Sec. 304.2.2 and 304.5.2 are inconsistent as G. F. Janssonius points out. Two basically different definitions have been in common use. One is based on "total elongation," the other on "permanent strain." When the ASTM group submitted specifications for stress relieved wire in which the definition of 1 per cent elongation was adopted, Sec. 304.2.2 was changed accordingly, but the definition of "0.2 per cent permanent strain" remained unchanged in 304.5.2.

LATERAL LOAD ANALYSIS OF TWO-COLUMN BENTS[a]

Discussion by B. R. Cooke

B. R. COOKE,[1] J.M. ASCE.—The author is correct in his statement that the usual shear distribution method is not too effective where the columns are very stiff as compared to the stiffness of the girders. However a simple and direct moment distribution method applicable for the solution of two-column symmetrical bents is given in a paper by L. E. Grinter and C. H. Tsao.[2] This cantilever moment distribution allows the structure to translate and the joints to rotate without any change in the column shear.

In Figure 1, a column is allowed to translate due to an applied moment. The angle change caused by this moment is $\frac{ML}{EI}$ as compared to $\frac{ML}{4EI}$ had the column not been allowed to translate at the top. Hence, for cantilever moment distribution which is based on a constant column shear, the effective stiffness of the column is only 1/4 that of a member in ordinary moment distribution. The carryover factor is -1, compared to +0.5 for moment distribution without joint translation (Positive moments are defined as those which tend to rotate the joint clockwise).

A solution of a simple bent using this cantilever moment distribution is given in Figure 2. The fixed-end moment for translation without rotation is +70 ft. kips at each end of each column (i.e., a total of 14 kips times 20 ft). The top member has an effective stiffness of 6EK due to the antisymmetrical moments caused by sidesway. The column has an effective stiffness of 1/4 (4EK) or EK. The distribution factors at Joint A are 1/7 for the column and 6/7 for the girder. The unbalanced moment of +70 causes distributions of -1/7(70) = -10 to the column and -6/7(70) = -60 to AA (the girder). The carryover to B is (-1)(-10) or +10. The total moments shown are the final moments.

For a further example the bent given in the author's Figure 1 is reduced to the top three stories as is shown in Figure 3. For the solution of the frame in Figure 3 fixed-end moments for translation without rotation are used. The girders have effective stiffnesses of 3/2 times their K values. The columns as cantilevers have 1/4 their K values as their effective stiffnesses. The solution of the frame is given in Figure 4.

A solution of the author's complete problem (author's Figure 1) by the cantilever method required ten cycles of distribution to obtain values of joint moments that were within 5% of the final values. Thus, although cantilever

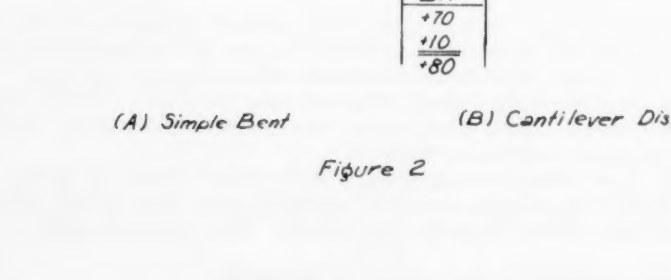
a. Proc. Paper 1638, May, 1958, by John E. Goldberg.

1. Resident Engr., Texas Highway Dept., Monahans, Tex.

2. "Joint Translation by Cantilever Moment Distribution" by L. E. Grinter and C. H. Tsao, Proceedings, ASCE, Vol. 79, Separate No. 79.

(A) Column (B) Moment Diagram

Figure 1



(A) Simple Bent (B) Cantilever Distribution

Figure 2

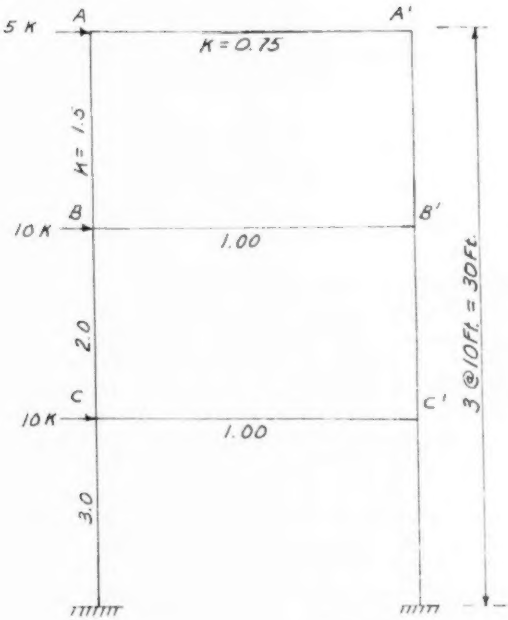


Figure 3

moment distribution for this type of problem is not rapid in closing, it has the advantages of being direct and of requiring little skill once one has become familiar with the simple modifications in the standard procedure for moment distribution. It appears to offer a simpler solution than that presented by the author and one at least as rapid.

A — A'

AB	AA'
.231	.769
+15.0	
-3.5	-11.5
+9.4	
-2.2	-7.2
+4.2	
-1.0	-3.2
+.7	
-.2	-.5
+.3	
-.1	-.2
+22.6	-22.6

B — B'

BA	BC	BB'
.157	.211	.632
+15.0	+45.0	
-9.4	-12.7	-37.9
+3.5	+23.0	
-4.2	-5.6	-16.7
+2.2	+2.5	
-0.7	-1.0	-3.0
+1.0	+1.1	
-.3	-.4	-1.4
+.2	+.2	
-.1	-.1	-.2
+7.2	+52.0	-59.2

C — C'

CB	CD	CC'
.192	.231	.577
+45.0	+75.0	
-23.0	-27.7	-69.2
+12.7		
-2.5	-2.9	-7.3
+5.6		
-1.1	-1.3	-3.2
+1.0		
-.2	-.2	-.6
+.4		
-.1	-.1	-.2
+37.8	+42.8	-80.5

D

DC
+75.0
+27.7
+2.9
+1.3
+.2
+107.1

Figure 4

WIND FORCES ON STRUCTURES: NATURE OF THE WIND[a]

Discussion by Arthur N. Gilbert

ARTHUR N. GILBERT.[1]—Local wind velocities for determining design wind loadings are illustrated on the map in Fig. 1. This information is very useful for structures in the U.S.A. but wind-velocity information is needed for areas outside of the United States.

The various staggered lines in Fig. 2A should be labeled or identified. The data shown in Fig. 2B should be placed opposite the related graph in Fig. 3 as shown on page 1708-13. The reference (in Table 1) to Fig. 1 probably should refer to Fig. 2A. On page 1708-12 in the last sentence of paragraph numbered 6, the figure reference should be changed from 3 to 2A.

In presenting his recommendations, I wonder if the author would want to comment on the design usefulness of "wind roses"? These wind roses or diagrams show the usual variation of wind velocity with direction and the direction of prevailing winds. These wind roses are usually found on plot plans of airports and are used to orient the runways. Could these diagrams be employed as structural design criteria in areas where detailed weather data is not readily available?

a. Proc. Paper 1708, July, 1958, by Robert H. Sherlock.

1. Asst. Sr. Engr., Socony-Mobil Oil Co., N.Y., N.Y.

WIND FORCES ON STRUCTURES: FUNDAMENTAL CONSIDERATIONS[a]

Discussion by Arthur N. Gilbert

ARTHUR N. GILBERT.[1]—In this paper the aerodynamic forces on structural members of uniform section are presented. The authors may wish to add their comments on the application of the design theory to structural members with varying cross section such as tapered stacks, street light stanchions, flag poles, etc.

a. Proc. Paper 1709, July, 1958, by Glen B. Woodruff and John J. Kozak.
1. Asst. Sr. Engr., Socony-Mobil Oil Co., N.Y., N. Y.

WIND FORCES ON STRUCTURES: FORCES ON ENCLOSED STRUCTURES[a]

Discussion by Arthur N. Gilbert

ARTHUR N. GILBERT.[1]—On page 1710-18 in the last sentence of paragraph D on Iowa Tests a reference is made to plotted data on Fig. 1. I cannot identify or find this data on the graph.

On page 1710-20 the term "Unit Overpressure" at the top of the page is less familiar than Gage Pressure which is another name for the same unit of measurement.

On page 1710-21 the symbol F_k at the bottom of the page could be described under terms as Reciprocating Lift Factor and the definition could be expressed in words as Force acting normal to wind direction.

On page 1710-22 the symbol f_s should be changed since this symbol in the American Standards Association abbreviations indicates a steel stress in reinforced concrete design.

The author may wish to comment on man-made wind forces or bomb blast effects. Some information on this subject is covered in "Blast - Resistant Concrete Houses; Design Considerations," 1956, Portland Cement Association, 33 West Grand Avenue, Chicago 10, Illinois.

The author may also wish to add to his bibliography a booklet titled "Windstorm Damage Prevention," National Board of Fire Underwriters, 85 John Street, New York 38, New York.

In this paper it may have been worthwhile to summarize the basis for establishing the effective wind load for design purposes. It could be stated this way:

Design wind pressure = velocity pressure x gust factor x shape factor x height factor

where: velocity pressure $= q = \dfrac{m\,V^2}{2\,g} = \dfrac{0.07651\ V^2}{2 \times 32.2} \left(\dfrac{5280}{3600}\right)^2$

V = wind velocity in miles per hour (M.P.H.) at 30 feet above grade

gust factor = 1.3 The gust factor is sometimes squared since it is a function of wind velocity.

shape factor = 1.3 for flat surface normal to the wind direction.

Note: For a stack, chimney or cylindrical tower care must be taken to determine if the shape factor is related to a datum of 1.0 or 1.3. In the current ASA Standard A 58.1 the shape factor is 0.6 for

a. Proc. Paper 1710, July, 1958, by Thomas H. Singell.

1. Asst. Sr. Engr., Socony-Mobil Oil Co., N. Y., N. Y.

cylindrical shapes and is derived from 0.8 ÷ 1.3. The 0.8 is the empirical shape factor with a datum of 1.0 whereas the ASA value is related to the formula for pressure on flat surfaces which involves the 1.3 quantity.

height factor = change in wind velocity with height according to the one-seventh power law.

$$\frac{V_h}{V_{30}} = \left(\frac{h}{30}\right)^{\frac{1}{7}}$$

And since the wind pressure varies with the square of the velocity

$$V_h^2 = \left(V_{30}\right)^2 \left(\frac{h}{30}\right)^{2/7}$$

where V_h = velocity of wind at an elevation of "h" feet above grade.

V_{30} = velocity of wind at 30 feet above grade.

Thus the general formula for design wind pressure "P" on a flat surface is as follows:

$$P = \frac{0.07651}{2 \times 32.2} \left(\frac{5280}{3600}\right)^2 \times 1.3 \times 1.3 \left(V_{30}\right)^2 \left(\frac{h}{30}\right)^{2/7}$$

$$P = 0.00433 \left(V_{30}\right)^2 \left(\frac{h}{30}\right)^{2/7}$$

WIND FORCES ON STRUCTURES: STRUCTURES SUBJECT TO OSCILLATION[a]

Discussions by Arthur N. Gilbert and Louis Balog

ARTHUR N. GILBERT.[1]—The author may wish to add to his bibliography a very good paper titled "The Design and Comparative Costs for High Stacks" by E. J. Stankiewicz, Paper No. 54-A-260, Dec. 1954, Power Division of the American Society of Mechanical Engineers.

LOUIS BALOG.[2]—Among the structures subject to oscillations discussed by the author the suspended roof structures were omitted. Allegedly, a number of such roofs exhibited vibrations and flutter phenomena to damaging extent and vertical and diagonal stay ropes were provided for their stabilization. Data concerning the properties and behavior of these roofs should have been included in this paper.

Only two cases of tied arch bridge hanger failures and excessive vibrations are known. One occured in 1927 the other in 1955. Considering the large number of bridges of this type, such defects have been very rare, because the use of flexible hangers was recommended more than sixty years ago and also because the failure of rigid hangers in 1927 warned against their unnecessary use. Hanger shapes were of great variety, like flat bars, solid, laced and batten plated sections, small closed box and pipe sections, early flexible hangers were cross shaped, built from four short legged angles, during the past three decades round bar and wire rope hangars were the most frequently used.

Stability computations of arch chords considered hanger rigidity due to sectional properties and due to stress since the past century, and as parts of the chord supporting lateral frames, hangers had much more rigid sections than when they acted as purely tension members. Occasionally, the latter were hinged at the top, or at both ends. These historical facts indicate that without reference to aerodynamic phenomena, experience with the behavior of hangers lead to the suitably most rigid or the most flexible solutions, according to structural function rather long ago, and excessive hanger vibrations reoccured only because the facts of experience were disregarded.(1)

Incorrect conclusions might be drawn from the author's statements concerning the safety of actual suspension bridges. Although it is indicated on page 1712-4 that "depending on the stiffness inherent in the structure" aerodynamic response is modified, it does not transpire from the paper that reasonable structural rigidities have unconditionally reduced the aerodynamic

a. Proc. Paper 1712, July, 1958, by F. B. Farquharson.
1. Asst. Sr. Engr., Socony-Mobil Oil Co., N.Y., N.Y.
2. Cons. Engr., Binghamton, N.Y.

effects to practical insignificance. Statistical data depict the actual conditions.

Among 2,000 suspension bridges described in various publications only 11 failed in wind during 122 years.* All these bridges were practically unstiffened. The spans varied from 260 ft to 2,800 ft, the width from 4 ft to 39 ft. The rather incomplete records indicate that all the longer spans failed in the one noded torsional mode like the Tacoma Narrows Bridge in 1940.

Aerodynamic instability was reported in 1924 in the description of the reconstruction of a Danube River Bridge, practically unstiffened by wooden trusses. It "was so flexible as to develop alarming oscillations under strong wind."(2) In November 1938 and January 1939, respectively, it was made public that two bridges, practically unstiffened by plate girders, were oscillating in wind. The great importance of the torsional rigidity of the suspended structure was demonstrated by the failure of the Tacoma Narrows Bridge in 1940, nevertheless, two oscillating bridges were built in 1943 with truss suspended structures of large vertical and no torsional stiffness. In November 1945 it was announced that any double lateral system which would satisfactorily limit torsional oscillations would be stressed beyond the breaking point by live load deflections. In April 1954, however, it was reported that a bottom lateral system, containing less than 2 percent of the suspended steel, increased the torsional frequency and the critical velocity 2.76 times. In August 1946 it was made public that all girder type sections were always unstable and that all aerodynamic properties of the truss suspended structure are determined by the shape of the chord at the deck level, portions of the suspended structure other than this chord have little or no effect on aerodynamic stability. In October 1948, the decisive importance of the shape of this windward chord and of a certain structural detail was reaffirmed. In June 1954 it was disclosed that this all important detail was a thin leading edge, achieved by the use of a trussed outside stringer.

Concerning typical girder stiffened bridges the author states that if h/b falls between 0.23 and 0.07, torsional oscillations with a node at midspan will catastrophically develop at relatively low wind velocity. Where $h/b > 0.23$, the bending mode is catastrophic. If $h/b < 0.07$, flutter may reach destructive amplitude in only a few oscillations from rest. Truss stiffened bridges, the author states, are subject to flutter if the torsional stiffness is inadequate.

What the adequate stiffnesses are, which make the aerodynamic effects sufficiently insignificant, the author failed to state. Likewise, published model results never defined the minimum required rigidities of the suspended structure, vertical and torsional, as demonstrated by the behavior of actual bridges. The model of heavy, short-span, unavoidably very rigid self-anchored bridge, $H/l^2 = 10.72$, $I = 0.0182\ l^2$, $I_t = 0.009$, $h/b = 0.15$, showed catastrophic torsional oscillation and a peak vertical amplitude in 31.6 mph wind. In contrast, the model of a long-span suspension bridge, $H/l^2 = 1.68$, $I = 0.0045\ l^2$, $I_t = 0.002\ l^2$, $h/b = 0.56$, indicated stability, with deck closed, up to 632 mph, and with 0.35 b wide open grating at the center and 0.08 b wide openings between the trusses and the deck it remained stable up to infinite mph velocity wind. It is evident that the minimum sufficient values of the vertical and torsional rigidities of the actual suspended structures

*A much larger number of pin-connected truss spans collapsed during the same period.

these models did not establish. The models are insensitive for the variation of expensive structural characteristics, and based exactly on the same data, very laborious frequency computations may differ more than 10 percent only because of the method used. Furthermore, expensive differences in structural quantities may result in practically identical frequencies. The critical wind velocities are influenced also by other more uncertain quantities to such an extent that the results of elaborate computations do not define the minimum required rigidities consistently with the accuracy of statical computations.

The author adopted the design criteria of the 1952 Report of the Advisory Committee. The recommended model wind conditions, established on the basis of a few local observations at the site of a proposed bridge, do not seem to help. Wind data and the corresponding movements of an actual bridge, collected for years, failed to result in model behavior even approximately similar to that of the prototype. The recommendation to improve the aerodynamic behavior of a long-span bridge by increasing its weight, because the beneficial effect of such increase is greater than suggested by the percentage increase of the weight of the bridge, has been contradicted by long-span bridges in any condition and on every occasion. In addition, such a measure is economically impossible for long-spans.

The model of a 3,500 ft span bridge, having $H/l^2 = 6.19$, indicated catastrophic torsion at 61 mph wind and high velocity flutter which precipitates strong torsional oscillation at considerably lower wind velocity. The model of a 3,800 ft span bridge, having $H/l^2 = 1.68$, indicated stability up to infinite mph wind velocity ($I = 0.0045\ l^2$, $I_t = 0.002\ l^2$). The relative weight of the latter bridge is 3.7 times smaller than of the former, which the model showed to be dangerously unstable despite its immense weight. The indicated rigidities of the suspended structure of the very much lighter bridge cost but a small fraction of the expense of only doubling its weight. The square of the span coefficients of the suspended structure of the bridge having $H/l^2 = 6.18$ can be smaller than of the bridge having $H/l^2 = 1.68$, but not as much smaller as suggested by the difference in their weights. Conditions are diametrically the opposite to those stated by the Advisory Committee. This is why the sum of the stiffness parameters of the cable and the truss, extended by the first harmonic coefficients, was proved not to be the measure of aerodynamic rigidity by the behavior of actual suspension bridges.

Doubling the mass of the bridge by adding its weight concentrated along the center line of the suspended structure of unchanged rigidities, Y. Rocard computed a $\sqrt{3}$ times larger critical velocity, or 72 percent increase for 100 percent increase in weight, contradicting the Advisory Committee.(3)

Wind-tunnel results were much less effected by section-model configuration in Europe than in the United States. In making comparative tests with simplified sections and truss and girder sections reproduced in true detail, A. Selberg found that the differences between the simplified and true models must be considerable before having a marked effect on the critical velocities.(4)

German design practice disregarded the recommendations of the Advisory Committee in building girder stiffened bridges exclusively during recent years. Examples are: Köln-Rodenkirchen $h/b = 0.13$, $I = 0.0042\ l^2$, $H/l^2 = 6.44$; Köln-Mülheim $h/b = 0.17$, $I = 0.0049\ l^2$, $H/l^2 = 5.74$; Mettlach $h/b = 0.22$, $I \sim 0.005\ l^2$, $H/l^2 = 9.87$; Passau $h/b = 0.29$, $I \sim 0.005\ l^2$, $H/l^2 = 9.55$; Whelen $h/b = 0.11$, $I = 0.0006\ l^2$,

$H/l^2 = 10.45$, this is a "Norwegian" type suspended structure, exhibited slight vibrations due to traffic despite its very rigid cables. None of these bridges made noticeable movements due to wind. The behavior of girder stiffened bridges was similar in the United States, as disclosed by their square of the span coefficients, as follows: Maumee River $h/b = 0.17$, $I = 0.0043\ l^2$, $H/l^2 = 8.38$; Bronx-Whitestone $h/b = 0.15$, $I = 0.0005\ l^2$, $H/l^2 = 3.44$; Thousand Islands $h/b = 0.20$, $I = 0.00039\ l^2$, $H/l^2 = 2.5$; Deer Isle $h/b = 0.28$, $I = 0.00036\ l^2$, $H/l^2 = 1.39$; Tacoma 1940 $h/b = 0.21$, $I = 0.00016\ l^2$, $H/l^2 = 1.53$. The square of the span coefficients of the Maumee River Bridge are similar to the German bridges in its span range, it was not reported moving. The oscillating Beauharnois Bridge in Canada, $h/b = 0.25$, $I = 0.0027\ l^2$, $H/l^2 = 3.7$, has only half as large square of the span coefficients as the German bridges in its span range and sectional arrangement.

Based on wind-tunnel tests and piezometric measurements of natural wind, Selberg recommends that 111 mph critical velocity be specified.(4) Section-model tests of the Köln-Rodenkirchen Bridge resulted in 124 mph critical velocity.(5) Such results do not seem to define design more precisely than properly interpreted square of the span coefficients. This has been admitted repeatedly, by the publication of critical velocities, or some frequencies, instead of the vertical and torsional moment of inertias used, the dead load and structural dimensions. More than was achieved during the past 18 years, can hardly be expected from model studies. The complete picture of the movements of actual bridges, however, may yield useful results. Likewise, experimentation with actual structures.

Suspended structures of various configurations are aerodynamically different, however, structural rigidities, of reasonable cost, make the aerodynamic excitations practically insignificant. Inherent rigidities of comparatively shallow box and prestressed suspended structures can safely overcome aerodynamic effects, and if anchorage conditions are favorable, suspension bridges can be economical in all span ranges. Nevertheless, anchorage research has been neglected.

Only a very few, practically unstiffened, bridges failed in wind, and only a very few not properly stiffened bridges caused trouble. Contrary to its very purpose, aerodynamic research in the United States hindered the reasonably more widespread use of suspension bridges.

REFERENCES

1. "Mississippi River Bridge at Dubuque, Iowa," Discussion by Louis Balog, Trans. ASCE, v.114, 1949, p. 1310.

2. "Rebuilding the First Budapest Suspension Bridge," by Joseph Beke, Engineering News-Record, June 26, 1924, p. 1096.

3. "L'Instabilite en mecanique," by Y. Rocard, Paris 1954, Translated by M.L. Meyer, London 1957, p. 170.

4. "Aerodynamic Stability of Suspension Bridges," by Arne Selberg, Pub. IABSE, Vol. 17, 1957, p. 213.

5. "Windkanalversuche über die Sicherheit gegen winderregte Schwingungen bei der Hängebrücke Köln-Rodenkirchen," by Rodulf Barbré and Rolf Ibing, Der Stahlbau, Vol. 27, 1958, p. 169.

PROCEEDINGS PAPERS

The technical papers published in the past year are identified by number below. Technical-division sponsorship is indicated by an abbreviation at the end of each Paper Number, the symbols referring to: Air Transport (AT), City Planning (CP), Construction (CO), Engineering Mechanics (EM), Highway (HW), Hydraulics (HY), Irrigation and Drainage (IR), Pipeline (PL), Power (PO), Sanitary Engineering (SA), Soil Mechanics and Foundations (SM), Structural (ST), Surveying and Mapping (SU), and Waterways and Harbors (WW), divisions. Papers sponsored by the Department of Conditions of Practice are identified by the symbols (PP). For titles and order coupons, refer to the appropriate issue of "Civil Engineering." Beginning with Volume 82 (January 1956) papers were published in Journals of the various Technical Divisions. To locate papers in the Journals, the symbols after the paper number are followed by a numeral designating the issue of a particular Journal in which the paper appeared. For example, Paper 1859 is identified as 1859 (HY 7) which indicates that the paper is contained in the seventh issue of the Journal of the Hydraulics Division during 1958.

VOLUME 83 (1957)

DECEMBER: 1449(HY6), 1450(HY6), 1451(HY6), 1452(HY6), 1453(HY6), 1454(HY6), 1455(HY6), 1456(HY6)c, 1457(PO6), 1458(PO6), 1459(PO6), 1460(PO6)c, 1461(SA6), 1462(SA6), 1463(SA6), 1464(SA6), 1465(SA6), 1466(SA6)c, 1467(AT2), 1468(AT2), 1469(AT2), 1470(AT2), 1471(AT2), 1472(AT2), 1473(AT2), 1474(AT2), 1475(AT2), 1476(AT2), 1477(AT2), 1478(AT2), 1479(AT2), 1480(AT2), 1481(AT2), 1482(AT2), 1483(AT2), 1484(AT2), 1485(AT2)c, 1486(BD2), 1487(BD2), 1488(PO6), 1489(PO6), 1490(BD2), 1491(BD2), 1492(HY6), 1493(BD2).

VOLUME 84 (1958)

JANUARY: 1494(EM1), 1495(EM1), 1496(EM1), 1497(IR1), 1498(IR1), 1499(IR1), 1500(IR1), 1501(IR1), 1502 (IR1), 1503(IR1), 1504(IR1), 1505(IR1), 1506(IR1), 1507(IR1), 1508(ST1), 1509(ST1), 1510(ST1), 1511(ST1), 1512(ST1), 1513(WW1), 1514(WW1), 1515(WW1), 1516(WW1), 1517(WW1), 1518(WW1), 1519(ST1), 1520 (EM1)c, 1521(IR1)c, 1522(ST1)c, 1523(WW1)c, 1524(HW1), 1525(HW1), 1526(HW1)c, 1527(HW1).

FEBRUARY: 1528(HY1), 1529(PO1), 1530(HY1), 1531(HY1), 1532(HY1), 1533(SA1), 1534(SA1), 1535(SM1), 1536(SM1), 1537(SM1), 1538(PO1)c, 1539(SA1), 1540(SA1), 1541(SA1), 1542(SA1), 1543(SA1), 1544(SM1), 1545(SM1), 1546(SM1), 1547(SM1), 1548(SM1), 1549(SM1), 1550(SM1), 1551(SM1), 1552(SM1), 1553(PO1), 1554(PO1), 1555(PO1), 1556(PO1), 1557(SA1)c, 1558(HY1)c, 1559(SM1)c.

MARCH: 1560(ST2), 1561(ST2), 1562(ST2), 1563(ST2), 1564(ST2), 1565(ST2), 1566(ST2), 1567(ST2), 1568 (WW2), 1569(WW2), 1570(WW2), 1571(WW2), 1572(WW2), 1573(WW2), 1574(PL1), 1575(PL1), 1576(ST2)c, 1577(PL1), 1578(PL1)c, 1579(WW2)c.

APRIL: 1580(EM2), 1581(EM2), 1582(HY2), 1583(HY2), 1584(HY2), 1585(HY2), 1586(HY2), 1587(HY2), 1588 (HY2), 1589(IR2), 1590(IR2), 1591(IR2), 1592(SA2), 1593(SU1), 1594(SU1), 1595(SU1), 1596(EM2), 1597(PO2), 1598(PO2), 1599(PO2), 1600(PO2), 1601(PO2), 1602(PO2), 1603(HY2), 1604(EM2), 1605(SU1)c, 1606(SA2), 1607(SA2), 1608(SA2), 1609(SA2), 1610(SA2), 1611(SA2), 1612(SA2), 1613(SA2), 1614(SA2)c, 1615(IR2)c, 1616 (HY2)c, 1617(SU1), 1618(PO2)c, 1619(EM2)c, 1620(CP1).

MAY: 1621(HW2), 1622(HW2), 1623(HW2), 1624(HW2), 1625(HW2), 1626(HW2), 1627(HW2), 1628(HW2), 1629 (ST3), 1630(ST3), 1631(ST3), 1632(ST3), 1633(ST3), 1634(ST3), 1635(ST3), 1636(ST3), 1637(ST3), 1638(ST3), 1639(WW3), 1640(WW3), 1641(WW3), 1642(WW3), 1643(WW3), 1644(WW3), 1645(SM2), 1646(SM2), 1647 (SM2), 1648(SM2), 1649(SM2), 1650(SM2), 1651(HW2), 1652(HW2)c, 1653(WW3)c, 1654(SM2), 1655(SM2), 1656(ST3)c, 1657(SM2)c.

JUNE: 1658(AT1), 1659(AT1), 1660(HY3), 1661(HY3), 1662(HY3), 1663(HY3), 1664(HY3), 1665(SA3), 1666 (PL2), 1667(PL2), 1668(PL2), 1669(AT1), 1670(PO3), 1671(PO3), 1672(PO3), 1673(PL2), 1674(PL2), 1675 (PO3), 1676(PO3), 1677(SA3), 1678(SA3), 1679(SA3), 1680(SA3), 1681(SA3), 1682(SA3), 1683(PO3), 1684 (HY3), 1685(SA3), 1686(SA3), 1687(PO3), 1688(SA3)c, 1689(PO3)c, 1690(HY3)c, 1691(PL2)c.

JULY: 1692(EM3), 1693(EM3), 1694(ST4), 1695(ST4), 1696(ST4), 1697(SU2), 1698(SU2), 1699(SU2), 1700(SU2), 1701(SA4), 1702(SA4), 1703(SA4), 1704(SA4), 1705(SA4), 1706(EM3), 1707(ST4), 1708(ST4), 1709(ST4), 1710 (ST4), 1711(ST4), 1712(ST4), 1713(SU2), 1714(SA4), 1715(SA4), 1716(SU2), 1717(SA4), 1718(EM3), 1719 (EM3), 1720(SU2), 1721(ST4)c, 1722(ST4), 1723(ST4), 1724(EM3)c.

AUGUST: 1725(HY4), 1726(HY4), 1727(SM3), 1728(SM3), 1729(SM3), 1730(SM3), 1731(SM3), 1732(SM3), 1733 (PO4), 1734(PO4), 1735(PO4), 1736(PO4), 1737(PO4), 1738(PO4), 1739(PO4), 1740(PO4), 1741(PO4), 1742 (PO4), 1743(PO4), 1744(PO4), 1745(PO4), 1746(PO4), 1747(PO4), 1748(PO4), 1749(PO4).

SEPTEMBER: 1750(IR3), 1751(IR3), 1752(IR3), 1753(IR3), 1754(IR3), 1755(ST5), 1756(ST5), 1757(ST5), 1758(ST5), 1759(ST5), 1760(ST5), 1761(ST5), 1762(ST5), 1763(ST5), 1764(ST5), 1765(WW4), 1766(WW4), 1767(WW4), 1768(WW4), 1769(WW4), 1770(WW4), 1771(WW4), 1772(WW4), 1773(WW4), 1774(IR3), 1775 (IR3), 1776(SA5), 1777(SA5), 1778(SA5), 1779(SA5), 1780(SA5), 1781(WW4), 1782(SA5), 1783(SA5), 1784 (IR3)c, 1785(WW4)c, 1786(SA5)c, 1787(ST5)c, 1788(IR3), 1789(WW4).

OCTOBER: 1790(EM4), 1791(EM4), 1792(EM4), 1793(EM4), 1794(EM4), 1795(HW3), 1796(HW3), 1797(HW3), 1798(HW3), 1799(HW3), 1800(HW3), 1801(HW3), 1802(HW3), 1803(HW3), 1804(HW3), 1805(HW3), 1806 (HY5), 1807(HY5), 1808(HY5), 1809(HY5), 1810(HY5), 1811(HY5), 1812(SM4), 1813(SM4), 1814(ST6), 1815 (ST6), 1816(ST6), 1817(ST6), 1818(ST6), 1819(ST6), 1820(ST6), 1821(ST6), 1822(EM4), 1823(PO5), 1824 (SM4), 1825(SM4), 1826(SM4), 1827(ST6)c, 1828(SM4)c, 1829(HW3)c, 1830(PO5)c, 1831(EM4)c, 1832(HY5)c

NOVEMBER: 1833(HY6), 1834(HY6), 1835(SA6), 1836(ST7), 1837(ST7), 1838(ST7), 1839(ST7), 1840(ST7), 1841(ST7), 1842(SU3), 1843(SU3), 1844(SU3), 1845(SU3), 1846(SU3), 1847(SA6), 1848(SA6), 1849(SA6), 1850(SA6), 1851(SA6), 1852(SA6), 1853(SA6), 1854(ST7), 1855(SA6)c, 1856(HY6)c, 1857(ST7)c, 1858 (SU3)c.

DECEMBER: 1859(HY7), 1860(IR4), 1861(IR4), 1862(IR4), 1863(SM5), 1864(SM5), 1865(ST8), 1866(ST8), 1867 (ST8), 1868(PP1), 1869(PP1), 1870(PP1), 1871(PP1), 1872(PP1), 1873(WW5), 1874(WW5), 1875(WW5), 1876 (WW5), 1877(CP2), 1878(ST8), 1879(ST8), 1880(HY7)c, 1881(SM5)c, 1882(ST8)c, 1883(PP1)c, 1884(WW5)c, 1885(CP2)c, 1886(PO6), 1887(PO6), 1888(PO6), 1889(PO6), 1890(HY7), 1891(PP1).

c. Discussion of several papers, grouped by divisions.